

ABSORBING REPRESENTATIONS WITH RESPECT TO CLOSED OPERATOR CONVEX CONES

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ABSTRACT. We initiate the study of absorbing representations of C^* -algebras with respect to closed operator convex cones. We completely determine when such absorbing representations exist, which leads to the question of characterising when a representation is absorbing, as in the classical Weyl–von Neumann type theorem of Voiculescu. In the classical case, this was solved by Elliott and Kucerovsky who proved that a representation is nuclearly absorbing if and only if it induces a purely large extension. By considering a related problem for extensions of C^* -algebras, which we call the purely large problem, we ask when a purely largeness condition similar to the one defined by Elliott and Kucerovsky, implies absorption with respect to some given closed operator convex cone.

We solve this question for a special type of closed operator convex cone induced by actions of finite topological spaces on C^* -algebras. As an application of this result, we give K -theoretic classification for certain C^* -algebras containing a purely infinite, two-sided, closed ideal for which the quotient is an AF algebra. This generalises a similar result by the second author, S. Eilers and G. Restorff in which all extensions had to be full.

1. INTRODUCTION

The study of absorbing representations dates back, in retrospect, more than a century to Weyl [Wey09] who proved that any bounded self-adjoint operator on a separable Hilbert space is a compact perturbation of a diagonal operator. This implies that the only points in the spectrum of a self-adjoint operator which are affected by compact perturbation, are the isolated points of finite multiplicity. The result was improved by von Neumann [vN35] who showed that the compact operator in question may be chosen to be a Hilbert–Schmidt operator of arbitrarily small norm. An important corollary, which is often referred to as *the Weyl–von Neumann theorem*, is that any two bounded self-adjoint operators S and T on a separable Hilbert space are *approximately unitarily equivalent* if and only if they have the same spectrum with the same multiplicity in all discrete points. Approximate unitary equivalence means that there exists a sequence of unitaries (U_n) such that $U_n^* S U_n - T$ is compact for each n and tends to zero.

In 1970, Halmos [Hal70] published ten open problems in Hilbert space theory. One of these problems was whether it was possible to generalise Weyl’s theorem to normal operators. Shortly after, this question was answered in the positive, independently, by

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Sikonia [Sik72] and Berg [Ber71]. Consequently they also obtained a generalisation of the Weyl–von Neumann theorem, for normal operators instead of self-adjoint operators.

Another problem of Halmos was the following: can every operator on a separable, infinite dimensional Hilbert space \mathcal{H} be approximated in norm by reducible operators? In 1976, Voiculescu [Voi76] provided an affirmative answer to this question. In proving this, he made a ground breaking generalisation of the other Weyl–von Neumann type theorems. He showed that any two unital representations $\mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H})$, which are faithful modulo the compacts, are approximately unitarily equivalent, for *any* separable, unital C^* -algebra \mathfrak{A} . This is equivalent to saying that any such representation Φ is *absorbing*, i.e. for any unital representation Ψ , the diagonal sum $\Phi \oplus \Psi$ is approximately unitarily equivalent to Φ .

The progress of Sikonia and Berg, motivated Brown, Douglas and Fillmore [BDF73] to classify normal operators in the Calkin algebra up to unitary equivalence. They found that it was equivalent to classify unital *extensions* of certain commutative C^* -algebras by the C^* -algebra of compact operators \mathbb{K} . This led them to construct the semigroup $\text{Ext}(\mathfrak{A})$ which they construct first for separable, unital, commutative C^* -algebras \mathfrak{A} and later in [BDF77] for any separable, unital C^* -algebra \mathfrak{A} . It follows from Voiculescu’s Weyl–von Neumann type theorem that $\text{Ext}(\mathfrak{A})$ always has a zero element. In [Arv77], Arveson combined this with the dilation theorem of Stinespring [Sti55], to show that an extension of \mathfrak{A} by the compact operators has an inverse in $\text{Ext}(\mathfrak{A})$ if and only if the extension has a completely positive splitting. In particular, by the lifting theorem of Choi and Effros [CE76], $\text{Ext}(\mathfrak{A})$ is a group if \mathfrak{A} is separable and nuclear.

The theory of studying extensions was revolutionised by the work of Kasparov in [Kas80b] when he constructed the commutative semigroup $\text{Ext}(\mathfrak{A}, \mathfrak{B})$ generated by extensions of \mathfrak{A} by $\mathfrak{B} \otimes \mathbb{K}$. In [Kas80a] Kasparov generalises the Stinespring theorem, using Hilbert modules instead of Hilbert spaces, and proves that there *exists* a zero element in $\text{Ext}(\mathfrak{A}, \mathfrak{B})$ if every completely positive map from \mathfrak{A} to \mathfrak{B} is *nuclear*. By the exact same methods developed by Arveson, it follows that $\text{Ext}(\mathfrak{A}, \mathfrak{B})$ is a group when \mathfrak{A} is nuclear, and Kasparov also proves that this group is isomorphic to $KK^1(\mathfrak{A}, \mathfrak{B})$.

Say that a completely positive map ϕ from \mathfrak{A} to the multiplier algebra $\mathcal{M}(\mathfrak{B})$ is *weakly nuclear* if $b^*\phi(-)b: \mathfrak{A} \rightarrow \mathfrak{B}$ is nuclear for all $b \in \mathfrak{B}$. What Kasparov actually proves when showing the existence of the zero element in $\text{Ext}(\mathfrak{A}, \mathfrak{B})$, is that there is a unital representation $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$ which is weakly nuclear, and such that the Cuntz sum (sometimes referred to as the BDF sum) with any weakly nuclear, unital representation $\mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$ is approximately unitarily equivalent to Φ . Any Φ which satisfies this latter condition is called *nuclearly absorbing*, since it absorbs any weakly nuclear representation. However, where as Voiculescu’s Weyl–von Neumann type theorem says that *any* representation, which is faithful modulo the compacts, is absorbing, Kasparov’s result only shows that a somewhat small class of representations are nuclearly absorbing. Kasparov notes in [Kas80a] that in general, a representation $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$ which is faithful modulo $\mathfrak{B} \otimes \mathbb{K}$, is not necessarily nuclearly absorbing, and remarks that it would be desirable to find conditions which determine when a representation is nuclearly absorbing.

Kirchberg showed in [Kir94], exactly when the obvious generalisation of Voiculescu’s theorem holds: Let \mathfrak{B} be a σ -unital, stable C^* -algebra which is not \mathbb{K} . Then \mathfrak{B} is purely infinite and simple if and only if for any separable, unital C^* -algebra \mathfrak{A} , any unital representation $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ which is faithful modulo \mathfrak{B} , is nuclearly absorbing. This was

used by Kirchberg to prove, amongst other things, that any separable, exact C^* -algebra embeds in the Cuntz algebra \mathcal{O}_2 , and also played an important part in his classification of the Kirchberg algebras.

It was not until 2001 that Elliott and Kucerovsky [EK01] found general conditions which determine nuclear absorption. We say that an extension of separable C^* -algebras $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ is *purely large*, if for any $x \in \mathfrak{E} \setminus \mathfrak{B}$, the C^* -algebra $\overline{x^*\mathfrak{B}x}$ contains a stable C^* -subalgebra which is full in \mathfrak{B} . They prove that if \mathfrak{A} and \mathfrak{B} are separable, \mathfrak{A} is unital and \mathfrak{B} is stable, then a unital representation $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is nuclearly absorbing if and only if the extension generated by Φ is purely large. To simplify the purely large condition, the concept of a C^* -algebra having the *corona factorisation property* was introduced (see [Kuc06] and [KN06b]). The following was proved: let \mathfrak{B} be a separable, stable C^* -algebra. Then \mathfrak{B} has the corona factorisation property if and only if for any separable, unital C^* -algebra \mathfrak{A} , any unital, *full* representation $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is nuclearly absorbing. It turns out that many C^* -algebras of interest have the corona factorisation property, including every σ -unital C^* -algebra with finite nuclear dimension [Rob11] and all separable C^* -algebras which absorb the Jiang–Su algebra [KR14].

These Weyl–von Neumann type theorems have been used in the classification program to classify simple C^* -algebras as well as non-simple C^* -algebras. Rørdam [Rør97] used the Weyl–von Neumann theorem of Kirchberg to classify extensions of UCT Kirchberg algebras. Embarking on the same idea, the second author, Eilers and Restorff [ERR09] classified a large class of extensions of C^* -algebras, by applying the Weyl–von Neumann theorem of Elliott and Kucerovsky. A necessary condition for this method to work is that the extensions have to be *full*. However, fullness of an extension is a huge limitation on the primitive ideal space of the extension algebra, and thus only very specific non-simple C^* -algebras can be classified in this way. The main motivation for this paper, was to generalise the Weyl–von Neumann theorem of Elliott and Kucerovsky in order to apply this to classify extensions which are *not* necessarily full. We found that the right way of doing this, was to consider a much more general problem using *closed operator convex cones*.

Closed operator convex cones were introduced by Kirchberg and have been important in the study of non-simple C^* -algebras, see e.g. [KR05], [KR02] and [Kir06]. A closed operator convex cone \mathcal{C} is a collection of completely positive maps $\mathfrak{A} \rightarrow \mathfrak{B}$ satisfying certain conditions (see Definition 2.1). Usually \mathcal{C} will be induced from an action of a topological space X on \mathfrak{A} and \mathfrak{B} , and will often consist only of maps which are nuclear in some suitable generalised sense. An example, which should be thought of as “the classical case” is when \mathcal{C} consists exactly of all nuclear maps from \mathfrak{A} to \mathfrak{B} . In this case one has chosen a trivial action of X to construct the closed operator convex cone. We say that a completely positive map $\phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is *weakly in* \mathcal{C} if $b^*\phi(-)b$ is in \mathcal{C} for all $b \in \mathfrak{B}$. In the classical case, this corresponds to a map being weakly nuclear. If \mathfrak{B} is stable, it makes sense to say that a representation $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is \mathcal{C} -*absorbing*, if it absorbs any representation which is weakly in \mathcal{C} . Again, the classical case corresponds to Φ being nuclearly absorbing. Two natural questions arise: do there exist any \mathcal{C} -absorbing representations which are themselves weakly in \mathcal{C} ? And is there a way of determining which representations are \mathcal{C} -absorbing, i.e. can we obtain Weyl–von Neumann type theorems for \mathcal{C} -absorbing representations?

The first of these questions is completely answered in Theorem 3.14 and relies only on whether or not \mathcal{C} is countably generated as a closed operator convex cone. If \mathfrak{A} and \mathfrak{B}

are both separable, then any closed operator convex cone is countably generated. Thus it remains to find Weyl–von Neumann type theorems which determine \mathcal{C} -absorption. By considering \mathcal{C} -purely largeness, a generalised version of purely largeness which takes the closed operator convex cone into consideration, we obtain what we refer to as *the purely large problem*. The problem basically is: can we determine classes of closed operator convex cones \mathcal{C} , for which a representation $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is \mathcal{C} -absorbing if and only if the induced extension is \mathcal{C} -purely large. We solve this problem (Theorem 6.11) for closed operator convex cones induced by actions of *finite* topological spaces X . In these cases, we also obtain Weyl–von Neumann type theorems resembling those of Kucerovsky (Theorem 6.22) and of Kirchberg (Theorem 7.5).

Finally, we apply our Weyl–von Neumann type results to classify certain non-simple C^* -algebras. What is basically proven is, that if \mathfrak{A} is a C^* -algebra of real rank zero containing a two-sided, closed ideal \mathfrak{I} , such that $\mathfrak{A}/\mathfrak{I}$ is an AF algebra, and \mathfrak{I} falls into a class of separable, nuclear, purely infinite C^* -algebras with finite primitive ideal spaces which are strongly classified by a K -theoretic invariant, then \mathfrak{A} can be classified by a K -theoretic invariant. Note that we are *not* requiring that the extension $0 \rightarrow \mathfrak{I} \rightarrow \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{I} \rightarrow 0$ is full, thus allowing \mathfrak{A} to have any (finite) primitive ideal space. This generalises a similar result by the second author, Eilers and Restorff [ERR13b] where this extension had to be full. We use this result to classify all graph C^* -algebras \mathfrak{A} which contain a purely infinite two-sided, closed ideal \mathfrak{I} with $\text{Prim } \mathfrak{I}$ finite, such that $\mathfrak{A}/\mathfrak{I}$ is an AF algebra. In [ERR13a], the second author, Eilers and Restorff consider the classification of graph C^* -algebras for which the primitive ideal spaces have at most four points. Our method shows, that we have classification in the separated cases, where the two-sided, closed ideal is purely infinite and the quotient is AF.

The paper is divided up as follows. In Section 2 we prove many basic properties about closed operator convex cones, and in particular, we prove dilation theorems à la Kasparov and Stinespring. In Section 3 we prove all basic theorems regarding absorbing representations with respect to closed operator convex cones, including an existence theorem of such. In Section 4 we consider the purely large problem with respect to closed operator convex cones. In Section 5 we introduce and prove many basic properties about actions of topological spaces on C^* -algebras, including a generalised notion of fullness for completely positive X -equivariant maps. We also prove that residually X -nuclear maps satisfy a certain approximation property similar to the classical approximation property of nuclear maps. In Section 6 we prove that for a large class of X - C^* -algebras, when X is finite, we obtain a Weyl–von Neumann type theorem à la Elliott and Kucerovsky, which determines when representations are weakly residually X -nuclearly absorbing. We also obtain a version using the corona factorisation property, where we may replace our generalised purely large condition, with the much simpler condition that the representation is full in the X -equivariant sense. Section 7 contains a generalised version of Kirchberg’s Weyl–von Neumann type theorem, for C^* -algebras with a finite primitive ideal space. Using this, we obtain a nice classification theorem using the ideal-related KK^1 -classes for extensions by such C^* -algebras. Finally, in Section 8 we use the above result to obtain K -theoretic classification of certain C^* -algebras, where we can separate the finite and the infinite part.

1.1. Notation. As for general notation, we let $\mathcal{M}(\mathfrak{B})$ denote the multiplier algebra of the C^* -algebra \mathfrak{B} , $\mathcal{Q}(\mathfrak{B}) := \mathcal{M}(\mathfrak{B})/\mathfrak{B}$ the corona algebra, and π denote the quotient map

$\mathcal{M}(\mathfrak{B}) \rightarrow \mathcal{Q}(\mathfrak{B})$. Any other quotient map $\mathfrak{E} \twoheadrightarrow \mathfrak{A}$ of C^* -algebras will usually be denoted by p .

By \mathbb{K} we denote the C^* -algebra of compact operators on a separable, infinite dimensional Hilbert space.

As is common in the literature, we write *c.p. map* instead of completely positive map. We let $CP(\mathfrak{A}, \mathfrak{B})$ denote the cone of c.p. maps from \mathfrak{A} to \mathfrak{B} .

For a C^* -algebra \mathfrak{A} , we let \mathfrak{A}^\dagger denote the *forced unitisation* of \mathfrak{A} , i.e. we add a unit to \mathfrak{A} regardless if \mathfrak{A} already had a unit. If $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive c.p. map and \mathfrak{B} is unital, then we let $\phi^\dagger: \mathfrak{A}^\dagger \rightarrow \mathfrak{B}$ denote the map given by

$$\phi^\dagger(a + \lambda 1) = \phi(a) + \lambda 1_{\mathfrak{B}}, \quad \text{for } a \in \mathfrak{A}, \text{ and } \lambda \in \mathbb{C}.$$

It is well-known that ϕ^\dagger is a unital c.p. map, and that if ϕ is a $*$ -homomorphism, then ϕ^\dagger is a unital $*$ -homomorphism.

If $\mathfrak{e}: 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ is an extension of C^* -algebras, then the *unitised extension* \mathfrak{e}^\dagger is the short exact sequence $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E}^\dagger \xrightarrow{p^\dagger} \mathfrak{A}^\dagger \rightarrow 0$.

We write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. At times we will write $a \approx b$ to just mean that a and b can be approximated arbitrarily well. This will severely simplify notation in certain proofs. An example of this could be, that if \mathfrak{A} is a simple, purely infinite C^* -algebra, and $a, b \in \mathfrak{A}$ are non-zero, positive elements then there are $c \in \mathfrak{A}$ such that $a \approx c^*bc$. We write $a \in_\epsilon S$ to mean that there is an $x \in S$ such that $a \approx_\epsilon x$.

We will say that $s_1, s_2 \in \mathfrak{E}$ are \mathcal{O}_2 -isometries if s_1 and s_2 are isometries such that $s_1 s_1^* + s_2 s_2^* = 1_{\mathfrak{E}}$.

2. CLOSED OPERATOR CONVEX CONES AND DILATION THEOREMS

In this section we prove some very general things about closed operator convex cones. Given a closed operator convex cone in $CP(\mathfrak{A}, \mathfrak{B})$ we consider two pictures of representations with respect to closed operator convex cones; a Hilbert module picture where we study maps $\mathfrak{A} \rightarrow \mathbb{B}(E)$ for some (countably generated) Hilbert \mathfrak{B} -module E , and a multiplier algebra picture where we (mainly) study maps $\mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$. We show that we have nice Kasparov–Stinespring dilation type theorems in both cases.

Definition 2.1. Let \mathfrak{A} and \mathfrak{B} be C^* -algebras and let $CP(\mathfrak{A}, \mathfrak{B})$ denote the cone of all completely positive (c.p.) maps from \mathfrak{A} to \mathfrak{B} . A subset \mathcal{C} of $CP(\mathfrak{A}, \mathfrak{B})$ is called a (*matricially*) *operator convex cone* if it satisfies the following:

- (1) \mathcal{C} is a cone,
- (2) If $\phi \in \mathcal{C}$ and b in \mathfrak{B} then $b^* \phi(-) b \in \mathcal{C}$,
- (3) If $\phi \in \mathcal{C}$, $a_1, \dots, a_n \in \mathfrak{A}$, and $b_1, \dots, b_n \in \mathfrak{B}$ then the map

$$(2.1) \quad \sum_{i,j=1}^n b_i^* \phi(a_i^*(-) a_j) b_j$$

is in \mathcal{C} .

We equip \mathcal{C} with the point-norm topology, and say that it is a *closed* operator convex cone, if it is closed as a subspace of $CP(\mathfrak{A}, \mathfrak{B})$.

Given a subset $\mathcal{S} \subset CP(\mathfrak{A}, \mathfrak{B})$ we let $K(\mathcal{S})$ denote the smallest closed operator convex cone containing \mathcal{S} .

We will say that a closed operator convex cone \mathcal{C} is *countably generated* if there is a countable set \mathcal{S} , such that $\mathcal{C} = K(\mathcal{S})$.

We will mainly be considering closed operator convex cones.

Example 2.2. A c.p. map is called *factorable* if it factors through a matrix algebra by c.p. maps. The set $CP_{\text{fact}}(\mathfrak{A}, \mathfrak{B}) \subset CP(\mathfrak{A}, \mathfrak{B})$ of all factorable maps is an operator convex cone.

A c.p. map is called *nuclear* if it can be approximated point-norm by factorable maps, i.e. if it is in the point-norm closure of $CP_{\text{fact}}(\mathfrak{A}, \mathfrak{B})$. The set $CP_{\text{nuc}}(\mathfrak{A}, \mathfrak{B})$ of nuclear c.p. maps is a closed operator convex cone.

The following lemma is well-known by many, but the authors have not been able to find a proof in the literature. Thus a proof is also provided. The lemma mainly states, that the definition of nuclear maps given above, agrees with the classical notion of nuclear maps.

Lemma 2.3. *Let $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ be a nuclear map as defined in Example 2.2. Then ϕ may be approximated in point-norm by c.p. maps of the form $\rho \circ \psi$ where $\psi: \mathfrak{A} \rightarrow M_n$ is a contractive c.p. map and $\rho: M_n \rightarrow \mathfrak{B}$ is a c.p. map with $\|\rho\| \leq \|\phi\|$.*

Proof. Let $\psi_\alpha: \mathfrak{A} \rightarrow M_{n_\alpha}$ and $\rho_\alpha: M_{n_\alpha} \rightarrow \mathfrak{B}$ be nets of c.p. maps such that $\rho_\alpha \circ \psi_\alpha$ converges point-norm to ϕ . Start by assuming that \mathfrak{A} is unital. For each α , $\psi_\alpha(1)$ generates a corner in M_{n_α} , say M_{k_α} , such that $\rho_\alpha \circ \psi_\alpha = \rho_\alpha|_{M_{k_\alpha}} \circ \psi_\alpha|^{M_{k_\alpha}}$. Hence we may assume, without loss of generality, that $\psi_\alpha(1)$ generates all of M_{n_α} as a corner, i.e. that $x_\alpha := \psi_\alpha(1)$ is invertible. By letting $\psi_\alpha^0 := x_\alpha^{-1/2} \psi_\alpha(-) x_\alpha^{-1/2}$ and $\rho_\alpha^0 := \rho_\alpha(x_\alpha^{1/2}(-) x_\alpha^{1/2})$ we get $\rho_\alpha \circ \psi_\alpha = \rho_\alpha^0 \circ \psi_\alpha^0$. Thus we may in fact assume that ψ_α is unital, in particular $\|\psi_\alpha\| = 1$.

Since $\|\phi\| = \|\phi(1)\| \approx \|\rho_\alpha(\psi_\alpha(1))\| = \|\rho_\alpha(1)\| = \|\rho_\alpha\|$, we may perturb ρ_α slightly to obtain $\|\rho_\alpha\| = \|\phi\|$ and we still get an approximation of ϕ . This finishes the case when \mathfrak{A} is unital.

If \mathfrak{A} is not unital, let \mathfrak{A}^\dagger denote the unitisation of \mathfrak{A} . Let (a_λ) be an approximate identity (of positive contractions) in \mathfrak{A} . Construct a net of c.p. maps $\phi_\lambda: \mathfrak{A}^\dagger \rightarrow \mathfrak{B}$ given by $\phi_\lambda(a + \mu 1) = \phi(a_\lambda a a_\lambda + \mu a_\lambda^2)$. We have $\|\phi_\lambda\| = \|\phi_\lambda(1)\| = \|\phi(a_\lambda^2)\| \leq \|\phi\|$ and clearly $\phi_\lambda|_{\mathfrak{A}}$ converges point-norm to ϕ . Let $\psi_{\alpha,\lambda}: \mathfrak{A}^\dagger \rightarrow M_{n_\alpha}$ be the c.p. map given by $\psi_{\alpha,\lambda}(a + \mu 1) = \psi_\alpha(a_\lambda a a_\lambda + \mu a_\lambda^2)$. Clearly $\rho_\alpha \circ \psi_{\alpha,\lambda}$ converges point-norm to ϕ_λ and thus ϕ_λ is nuclear. By the unital case, ϕ_λ has the desired approximation, and so does the restriction to \mathfrak{A} which approximates ϕ . \square

The following is a classical example where the closed operator convex cone is generated by an action of a topological space. Many more examples, using the structure of X - C^* -algebras, will be provided in Remarks 5.2 and 5.24.

Example 2.4. Let \mathfrak{A} and \mathfrak{B} be continuous $C_0(X)$ -algebras over some locally compact Hausdorff space X . The set $CP(X; \mathfrak{A}, \mathfrak{B})$ of all $C_0(X)$ -linear c.p. maps is a closed operator convex cone.

Remark 2.5. If $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ is point-norm closed, then condition (3) implies condition (2) in Definition 2.1, by letting $n = 1$, $b = b_1$, and letting a_1 run through an approximate identity in \mathfrak{A} .

Moreover, if \mathcal{C} is a closed operator convex cone, $\phi \in \mathcal{C}$, $a_1, \dots, a_n \in \mathcal{M}(\mathfrak{A})$, and $b_1, \dots, b_n \in \mathcal{M}(\mathfrak{B})$, then by a similar argument as above, the map in equation (2.1) is

in \mathcal{C} , since the unit ball in any C^* -algebra is strictly dense in the unit ball of its multiplier algebra.

Remark 2.6. The point-norm closure of an operator convex cone is clearly a closed operator convex cone. Let $\mathcal{S} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a subset. It can easily be seen that the subset $C(\mathcal{S}) \subset CP(\mathfrak{A}, \mathfrak{B})$ consisting of finite sums of maps of the type in equation (2.1) with $\phi \in \mathcal{S}$, is an operator convex cone. In general $C(\mathcal{S})$ does *not* contain \mathcal{S} , however every map in \mathcal{S} is in the point-norm closure of $C(\mathcal{S})$. Hence $K(\mathcal{S})$ is the point-norm closure of $C(\mathcal{S})$. In other words, any map in $K(\mathcal{S})$ may be approximated point-norm by sums of maps of the form $\sum_{i,j=1}^n b_i^* \phi(a_i^*(-)a_j)b_j$ with $\phi \in \mathcal{S}$, $a_1, \dots, a_n \in \mathfrak{A}$ and $b_1, \dots, b_n \in \mathfrak{B}$.

Remark 2.7. Suppose that E is a right Hilbert \mathfrak{B} -module and that $\phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ is a c.p. map. Recall, e.g. from [Lan95, Chapter 5], that we may construct a right Hilbert \mathfrak{B} module $\mathfrak{A} \otimes_\phi E$ by taking the algebraic tensor product $\mathfrak{A} \odot E$ equipped with the semi-inner product induced by $\langle a \otimes x, b \otimes y \rangle = \langle x, \phi(a^*b)y \rangle_E$, quotient out length zero vectors, and taking the completion. We will abuse notation slightly and write $a \otimes x \in \mathfrak{A} \otimes_\phi E$ for the element induced by the elementary tensor $a \otimes x \in \mathfrak{A} \odot E$. There is a canonical $*$ -homomorphism $\omega: \mathfrak{A} \rightarrow \mathbb{B}(\mathfrak{A} \otimes_\phi E)$ given by left multiplication on the left tensor. We will often refer to this $*$ -homomorphism as the *dilating $*$ -homomorphism*.

Condition (3) in the above definition is easily seen to be equivalent to that $\langle y, \omega(-)y \rangle_{\mathfrak{A} \otimes_\phi B}$ is in \mathcal{C} for all $y = \sum_{i=1}^n a_i \otimes b_i \in \mathfrak{A} \otimes_\phi B$.

If \mathcal{C} is a point-norm closed subset of $CP(\mathfrak{A}, \mathfrak{B})$ then condition (3) is equivalent to that $\langle y, \omega(-)y \rangle_{\mathfrak{A} \otimes_\phi B}$ is in \mathcal{C} for all $y \in \mathfrak{A} \otimes_\phi B$.

The above remarks show, that closed operator convex cones behave nicely with respect to both multiplier algebras and Hilbert C^* -modules. We will first study the relations to Hilbert C^* -modules and use this to derive results in a multiplier algebra picture (for stable C^* -algebras).

2.1. The Hilbert C^* -module picture.

Definition 2.8. Let \mathfrak{A} and \mathfrak{B} be C^* -algebras and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Let E be a Hilbert \mathfrak{B} -module. A c.p. map $\phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ is said to be *weakly in \mathcal{C}* if the map

$$(2.2) \quad \mathfrak{A} \ni a \mapsto \sum_{i,j=1}^n \langle x_i, \phi(a_i^* a a_j) x_j \rangle_E \in \mathfrak{B}$$

is in \mathcal{C} for every $x_1, \dots, x_n \in E$ and $a_1, \dots, a_n \in \mathfrak{A}$.

Remark 2.9. By condition (3) above every $\phi \in \mathcal{C}$ is weakly in \mathcal{C} when considered as a map $\mathfrak{A} \rightarrow B \cong \mathbb{K}(B)$. Moreover, it is obvious that if E is a complemented Hilbert \mathfrak{B} -submodule of F , and $\phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ is weakly in \mathcal{C} , then the composition $\mathfrak{A} \xrightarrow{\phi} \mathbb{B}(E) \hookrightarrow \mathbb{B}(F)$ is weakly in \mathcal{C} .

Let $\mathcal{H}_{\mathfrak{B}} = \bigoplus_{n=1}^{\infty} \mathfrak{B}$, and let ι be the map $\mathfrak{B} \cong \mathbb{K}(\mathfrak{B}) \hookrightarrow \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ induced by embedding $\mathfrak{B} \hookrightarrow \mathcal{H}_{\mathfrak{B}}$ in the first coordinate. By the above, a c.p. map $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ is in \mathcal{C} if and only if $\iota \circ \phi$ is weakly in \mathcal{C} .

Proposition 2.10. *Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone, E be a Hilbert \mathfrak{B} -module, and $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ be a $*$ -homomorphism. Then Φ is weakly in \mathcal{C} if and only if $\langle x, \Phi(-)x \rangle_E$ is in \mathcal{C} for every $x \in E$.*

In particular, a c.p. map $\phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ is weakly in \mathcal{C} if and only if the dilating $$ -homomorphism $\omega: \mathfrak{A} \rightarrow \mathbb{B}(\mathfrak{A} \otimes_\phi E)$ is weakly in \mathcal{C} .*

Proof. The “if” part follows from the observation that

$$\sum_{i,j=1}^n \langle x_i, \Phi(a_i^* a a_j) x_j \rangle_E = \langle x, \Phi(a) x \rangle_E$$

for $x = \sum_{i=1}^n \Phi(a_i) x_i$. For the converse, suppose Φ is weakly in \mathcal{C} , let $x \in E$ and (a_λ) be an approximate identity in \mathfrak{A} . Then the maps $a \mapsto \langle x, \Phi(a_\lambda^* a a_\lambda) x \rangle_E$ are in \mathcal{C} and converge point-norm to $\langle x, \Phi(-)x \rangle_E$, which is thus in \mathcal{C} since \mathcal{C} is closed. \square

The above proposition, although of interest itself, also allows us to prove a Kasparov–Stinespring type theorem (c.f. [Kas80a]) for c.p. maps weakly in \mathcal{C} .

Remark 2.11. Let $\phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ be a c.p. map, and let ϕ_∞ denote the infinite repeat $\mathfrak{A} \rightarrow \mathbb{B}(E^\infty)$ where $E^\infty := E \oplus E \oplus \dots$. If \mathcal{C} is a closed operator convex cone and ϕ is weakly in \mathcal{C} , then ϕ_∞ is weakly in \mathcal{C} . In fact, let $\omega: \mathfrak{A} \rightarrow \mathbb{B}(\mathfrak{A} \otimes_\phi E)$ be the dilating $*$ -homomorphism which is weakly in \mathcal{C} by the above proposition. The infinite repeat ω_∞ of ω may be identified with the dilating $*$ -homomorphism of ϕ_∞ in a canonical way, so by the above proposition it suffices to show that ω_∞ is weakly in \mathcal{C} . If $(x_n) \in (\mathfrak{A} \otimes_\phi E)^\infty$ then $\langle (x_n), \omega_\infty(-)(x_n) \rangle = \sum_{n=1}^\infty \langle x_n, \omega(-)x_n \rangle$ is a point-norm limit of maps in \mathcal{C} , and is thus also itself in \mathcal{C} .

Recall that a Hilbert \mathfrak{B} -module E is called *full* if $\overline{\text{span}}\{\langle x, x \rangle : x \in E\} = \mathfrak{B}$.

In our Kasparov–Stinespring theorem below, there is both a non-unital and a unital version. However, there is a clear obstruction, given a closed operator convex cone \mathcal{C} , for when there can exist a unital completely positive map $\phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ weakly in \mathcal{C} . In fact, we must have that $\langle x, \phi(1)x \rangle = \langle x, x \rangle$ is a value obtained by some map in \mathcal{C} for every x . Thus if E is full then \mathcal{C} can not factor through any proper two-sided, closed ideal \mathfrak{J} in \mathfrak{B} , i.e. there is a $\psi \in \mathcal{C}$ such that $\psi(\mathfrak{A}) \not\subset \mathfrak{J}$. It turns out that this is the only obstruction, as can be seen in the Kasparov–Stinespring theorem below. The observation motivates the following definition.

Definition 2.12. Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. We say that \mathcal{C} is *non-degenerate* if for any proper two-sided, closed ideal \mathfrak{J} in \mathfrak{B} there is a map $\phi \in \mathcal{C}$ such that $\phi(\mathfrak{A}) \not\subset \mathfrak{J}$.

It is obvious that \mathcal{C} is non-degenerate if and only if the two-sided, closed ideal generated by $\{\phi(a) : a \in \mathfrak{A}, \phi \in \mathcal{C}\}$ is all of \mathfrak{B} .

In the case when \mathfrak{A} is σ -unital and $h \in \mathfrak{A}$ is strictly positive, then \mathcal{C} is non-degenerate if and only if the two-sided, closed ideal generated by $\{\phi(h) : \phi \in \mathcal{C}\}$ is all of \mathfrak{B} . This follows easily from the fact that any positive element in \mathfrak{A} can be approximated by a positive element of the form $(ah^{1/2})^*(ah^{1/2}) \leq \|a\|^2 h$.

Theorem 2.13 (Kasparov–Stinespring Theorem for Hilbert C^* -modules). *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras with \mathfrak{A} separable and \mathfrak{B} σ -unital, let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator*

convex cone, let E be a countably generated, full Hilbert \mathfrak{B} -module, and let $\phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ be a contractive completely positive map weakly in \mathcal{C} . Then there is a representation $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ weakly in \mathcal{C} and isometries $V, W \in \mathbb{B}(E, \mathcal{H}_{\mathfrak{B}})$ with $VV^* + WW^* = 1$, such that $V^*\Phi(-)V = \phi$.

If, in addition, \mathfrak{A} is unital and \mathcal{C} is non-degenerate, then there is a unital representation $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ weakly in \mathcal{C} and an element $V \in \mathbb{B}(E, \mathcal{H}_{\mathfrak{B}})$ such that $V^*\Phi(-)V = \phi$. If ϕ is unital, then such a V exists, for which there is another isometry W with $VV^* + WW^* = 1$.

Proof. The construction of Φ follows the original construction of Kasparov [Kas80a] very closely, however with subtle changes. Since we need this explicit construction in our proof we will repeat most of Kasparov's proof.

We may extend ϕ to a unital c.p. map $\phi^\dagger: \mathfrak{A}^\dagger \rightarrow \mathbb{B}(E)$ (see e.g. [BO08, Section 2.2]). Construct the Hilbert \mathfrak{B} -module $F = \mathfrak{A}^\dagger \otimes_{\phi^\dagger} E$, and let $\omega: \mathfrak{A} \rightarrow \mathbb{B}(F)$ be the $*$ -homomorphism which is left multiplication on the left tensor. Since \mathfrak{A} is separable, F is countably generated and it is clearly full (since $\langle 1 \otimes x, 1 \otimes x \rangle_F = \langle x, x \rangle_E$). Also, since \mathfrak{B} is σ -unital, it follows by [MP84, Theorem 1.9] that $F^\infty := F \oplus F \oplus \dots \cong \mathcal{H}_{\mathfrak{B}}$. Hence it suffices to prove the result for F^∞ in place of $\mathcal{H}_{\mathfrak{B}}$. Let $V, W \in \mathbb{B}(E, F^\infty)$ be given by

$$V(x) = (1 \otimes x, 0, 0, \dots), \quad W(x) = (0, 1 \otimes x, 1 \otimes x, \dots)$$

which have adjoints induced by

$$V^*(a \otimes x, y_2, y_3, \dots) = \phi^\dagger(a)x, \quad W^*(y_1, a_2 \otimes x_2, \dots) = \sum_{k=1}^{\infty} \phi^\dagger(a_k)x_k.$$

One should of course check that these in fact induce operators and that V and W are isometries for which $VV^* + WW^* = 1$. Now, let $\Phi = \omega_\infty: \mathfrak{A} \rightarrow \mathbb{B}(F^\infty)$ be the infinite repeat. Then clearly $V^*\Phi(-)V = \phi$. To see that Φ is weakly in \mathcal{C} , it suffices by Proposition 2.10 and Remark 2.11 to show that $\langle y, \omega(-)y \rangle$ is in \mathcal{C} for every $y \in F$. It suffices to check for elements of the form $y = \sum_{i=1}^n (a_i + \lambda_i 1) \otimes x_i$ where $x_i \in E$, $a_i \in \mathfrak{A}$ and $\lambda_i \in \mathbb{C}$. Letting (c_λ) be an approximate identity in \mathfrak{A} we get that

$$\begin{aligned} \langle y, \omega(a)y \rangle_F &= \sum_{i,j=1}^n \langle x_i, \phi^\dagger((a_i + \lambda_i 1)^* a (a_j + \lambda_j 1)^*) x_j \rangle_E \\ &= \lim_{\lambda} \sum_{i,j=1}^n \langle x_i, \phi((a_i + \lambda_i c_\lambda)^* a (a_j + \lambda_j c_\lambda)^*) x_j \rangle_E \end{aligned}$$

which is in \mathcal{C} as a function of a , since \mathcal{C} is closed. Thus Φ is weakly in \mathcal{C} .

Now suppose \mathfrak{A} is unital. If ϕ is unital, then a proof exactly as above (with \mathfrak{A} instead of \mathfrak{A}^\dagger) yields a unital representation Φ weakly in \mathcal{C} , and \mathcal{O}_2 -isometries V, W such that $V^*\Phi(-)V = \phi$. So it remains to prove the case where ϕ is not necessarily unital.

We will start by proving that there exists a unital representation $\Psi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ weakly in \mathcal{C} , by using that \mathcal{C} is non-degenerate. As above, it suffices to show that there is a unital representation $\Psi: \mathfrak{A} \rightarrow \mathbb{B}(G)$ weakly in \mathcal{C} where G is countably generated and full. In fact, then the infinite repeat will do the trick, since $G^\infty \cong \mathcal{H}_{\mathfrak{B}}$.

Fix $h \in \mathfrak{B}$ a strictly positive element. Since \mathcal{C} is non-degenerate, we may for each n find $\phi_n \in \mathcal{C}$ such that $h \in_{1/n} \overline{\mathfrak{B}\phi_n(1)\mathfrak{B}}$ (a priori we find a finite subset \mathcal{S}_n of \mathcal{C} such that $h \in_{1/n} \overline{\mathfrak{B}\{\phi'(1) : \phi' \in \mathcal{S}_n\}\mathfrak{B}}$, but then $\phi_n = \sum_{\phi' \in \mathcal{S}_n} \phi'$ also works). Letting $G_n = \mathfrak{A} \otimes_{\phi_n} \mathfrak{B}$,

we clearly get that $h \in_{1/n} \langle G_n, G_n \rangle$. Hence if $G := \bigoplus G_n$ then $h \in \langle G, G \rangle$ and thus G is full and also countably generated. Clearly there is an induced unital representation $\Psi: \mathfrak{A} \rightarrow \mathbb{B}(G)$ weakly in \mathcal{C} .

Let $F = \mathfrak{A} \otimes_\phi E$ which is countably generated, and $\omega: \mathfrak{A} \rightarrow \mathbb{B}(F)$ be the canonical $*$ -homomorphism which is weakly in \mathcal{C} by Proposition 2.10. By Kasparov's stabilisation theorem [Kas80a, Theorem 2] $F \oplus \mathcal{H}_{\mathfrak{B}} \cong \mathcal{H}_{\mathfrak{B}}$, so it suffices to prove the result for $F \oplus \mathcal{H}_{\mathfrak{B}}$ in place of $\mathcal{H}_{\mathfrak{B}}$. Let $\Psi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ be a unital representation weakly in \mathcal{C} , and let $\Phi = \omega \oplus \Psi: \mathfrak{A} \rightarrow \mathbb{B}(F \oplus \mathcal{H}_{\mathfrak{B}})$ which is a unital $*$ -homomorphism weakly in \mathcal{C} since both ω and Ψ are unital and weakly in \mathcal{C} . If $V \in \mathbb{B}(E, F \oplus \mathcal{H}_{\mathfrak{B}})$ is given by $V(x) = (1 \otimes x, 0)$ then $V^* \Phi(-) V = \phi$. \square

Corollary 2.14. *Let \mathfrak{A} be separable and unital, and \mathfrak{B} be σ -unital and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Then there exists a unital representation $\mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ weakly in \mathcal{C} if and only if \mathcal{C} is non-degenerate.*

2.2. The multiplier algebra picture. In order to obtain a similar Kasparov–Stinespring result for multiplier algebras, we need to construct new operator convex cones as below.

Lemma 2.15. *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone, and let E be a Hilbert \mathfrak{B} -module. Then $\mathcal{C}_E \subset CP(\mathfrak{A}, \mathbb{K}(E))$ given by*

$$\mathcal{C}_E := \{\phi \in CP(\mathfrak{A}, \mathbb{K}(E)) \mid \phi: \mathfrak{A} \rightarrow \mathbb{K}(E) \text{ is weakly in } \mathcal{C}\}$$

is a closed operator convex cone.

Moreover, a c.p. map $\phi: \mathfrak{A} \rightarrow \mathbb{K}(E)$ is weakly in \mathcal{C} if and only if $T^ \phi(-) T$ is in \mathcal{C}_E for every $T \in \mathbb{K}(E)$.*

Proof. \mathcal{C}_E is obviously point-norm closed, and clearly satisfies conditions (1) of Definition 2.1. Since \mathcal{C}_E is closed it suffices to check condition (3).

Let $\phi \in \mathcal{C}_E$, and let $a_1, \dots, a_n \in \mathfrak{A}$ and $T_1, \dots, T_n \in \mathbb{K}(E)$. We should show that

$$a \mapsto \Phi(a) := \sum_{i,j=1}^n T_i^* \phi(a_i^* a a_j) T_j$$

is weakly in \mathcal{C} . Let $x_1, \dots, x_m \in E$ and $c_1, \dots, c_m \in \mathfrak{A}$. Then

$$a \mapsto \sum_{k,l=1}^m \langle x_k, \Phi(c_k^* a c_l) x_l \rangle = \sum_{k,l=1}^m \sum_{i,j=1}^n \langle T_i x_k, \phi((c_k a_i)^* a (c_l a_j)) T_j x_l \rangle$$

is in \mathcal{C} , since ϕ is weakly in \mathcal{C} . Thus \mathcal{C}_E is a closed operator convex cone.

If ϕ is weakly in \mathcal{C} then $T^* \phi(-) T$ is weakly in \mathcal{C}_E for T since

$$\sum_{i,j=1}^n \langle x_i, T^* \phi(a_i^* a a_j) T x_j \rangle = \sum_{i,j=1}^n \langle T x_i, \phi(a_i^* a a_j) (T x_j) \rangle.$$

If conversely $T^* \phi(-) T$ is weakly in \mathcal{C}_E for every T , and (T_λ) is an approximate identity in $\mathbb{K}(E)$, then ϕ is weakly in \mathcal{C} since

$$\sum_{i,j=1}^n \langle x_i, \phi(a_i^* a a_j) x_j \rangle = \lim_{\lambda} \sum_{i,j=1}^n \langle x_i, T_\lambda^* \phi(a_i^* a a_j) T_\lambda x_j \rangle.$$

\square

In light of Remark 2.9 we observe, that if we identify \mathfrak{B} with $\mathbb{K}(\mathfrak{B})$, then $\mathcal{C}_{\mathbb{K}(\mathfrak{B})} = \mathcal{C}$. Thus, by Lemma 2.15, it makes sense to make the following definition.

Definition 2.16. Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. A c.p. map $\phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is said to be *weakly in* \mathcal{C} if $b^*\phi(-)b$ is in \mathcal{C} for every $b \in \mathfrak{B}$.

To be fair, this definition is the original definition. The name *weakly* is due to the fact, that $\phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is weakly in \mathcal{C} exactly when the composition $\mathfrak{A} \xrightarrow{\phi} \mathcal{M}(\mathfrak{B}) \hookrightarrow \mathfrak{B}^{**}$ is in the point-weak closure of \mathcal{C} in $CP(\mathfrak{A}, \mathfrak{B}^{**})$. The proof of this is a simple Hahn–Banach separation argument.

To obtain a multiplier algebra version of the Kasparov–Stinespring Theorem, we apply the version for Hilbert modules, and obtain a dilating $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$ which would be weakly in \mathcal{C}^s , where $\mathcal{C}^s \subset CP(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K})$ is the closed operator convex cone corresponding to $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$ when identifying $\mathfrak{B} \otimes \mathbb{K}$ with $\mathbb{K}(\mathcal{H}_{\mathfrak{B}})$. However, it seems more desirable, if \mathfrak{B} is stable, to obtain a dilating $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ which is weakly in \mathcal{C} .

First we will show a one-to-one correspondence between closed operator convex cones of $CP(\mathfrak{A}, \mathfrak{B})$ and $CP(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K})$ which preserves countably generated cones. In fact, we prove something more general.

Proposition 2.17. Let $\mathfrak{A}, \mathfrak{B}$ and \mathfrak{D} be C^* -algebras such that \mathfrak{D} is simple and nuclear. Then there is a bijection

$$\begin{aligned} \left\{ \begin{array}{c} \text{closed operator convex cones} \\ \text{in } CP(\mathfrak{A}, \mathfrak{B}) \end{array} \right\} &\leftrightarrow \left\{ \begin{array}{c} \text{closed operator convex cones} \\ \text{in } CP(\mathfrak{A}, \mathfrak{B} \otimes \mathfrak{D}) \end{array} \right\} \\ \mathcal{C} &\mapsto \mathcal{C} \otimes \mathfrak{D} := K(\{\phi(-) \otimes d : \phi \in \mathcal{C}\}) \\ \{(id_{\mathfrak{B}} \otimes \eta) \circ \psi : \psi \in \mathcal{K}, \eta \text{ is a state on } \mathfrak{D}\} &\leftarrow \mathcal{K}, \end{aligned}$$

where $d \in \mathfrak{D}$ is any fixed non-zero, positive element, and $id_{\mathfrak{B}} \otimes \eta: \mathfrak{B} \otimes \mathfrak{D} \rightarrow \mathfrak{B}$ is the slice map.

Moreover, if \mathfrak{D} is separable, then \mathcal{C} is countably generated if and only if $\mathcal{C} \otimes \mathfrak{D}$ is countably generated.

To prove Proposition 2.17 we will need the following amazing result of Kirchberg. The result is a part of [Kir03, Theorem 9.3]. We have included a proof for completion.

Theorem 2.18 (Kirchberg). Let $\mathcal{S} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a subset and $\phi \in CP(\mathfrak{A}, \mathfrak{B})$. Then $\phi \in K(\mathcal{S})$ if and only if for every positive $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$, the element $(id \otimes \phi)(c)$ is in the closed, two-sided ideal $\mathfrak{I}(c)$ of $C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{B}$ generated by

$$\{(id \otimes \psi)((1 \otimes a^*)c(1 \otimes a)) : \psi \in \mathcal{S}, a \in \mathfrak{A}\}.$$

Proof. “only if”: For any positive $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$ let $\mathcal{C}_c \subset CP(\mathfrak{A}, \mathfrak{B})$ be the set of all maps ψ such that $(id \otimes \psi)((1 \otimes a^*)c(1 \otimes a))$ is in $\mathfrak{I}(c)$. By polar decomposition of Hermitian forms, it follows that $(id \otimes \psi)((1 \otimes a^*)c(1 \otimes b)) \in \mathfrak{I}(c)$ for all $a, b \in \mathfrak{A}$ and $\psi \in \mathcal{C}_c$. Hence it easily follows that \mathcal{C}_c is a closed operator convex cone containing \mathcal{S} , and thus $K(\mathcal{S}) \subset \mathcal{C}_c$.

“if”: Suppose that $\phi \notin K(\mathcal{S})$. A standard Hahn–Banach argument implies that there are $a_1, \dots, a_n \in \mathfrak{A}$, $\epsilon > 0$ and f_1, \dots, f_n states on \mathfrak{B} such that for any $\psi \in K(\mathcal{S})$ we have $|f_i(\phi(a_i)) - f_i(\psi(a_i))| \geq \epsilon$ for some $i = 1, \dots, n$.

By [KR02, Lemma 7.17 (i)] there is a representation $\pi: \mathfrak{B} \rightarrow \mathbb{B}(\mathcal{H})$ with a cyclic vector ξ , and $c_1, \dots, c_n \in \pi(\mathfrak{B})'$ such that $f_i(b) = \langle \pi(b)c_i\xi, \xi \rangle$. Let $p: C^*(\mathbb{F}_\infty) \rightarrow C^*(1, c_1, \dots, c_n)$ be a $*$ -epimorphism and $d_1, \dots, d_n \in C^*(\mathbb{F}_\infty)$ be such that $p(d_i) = c_i$. For any $\phi' \in CP(\mathfrak{A}, \mathfrak{B})$ we may construct a positive linear functional $\rho_{\phi'}$ on $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}$ given on elementary tensors by $\rho_{\phi'}(d \otimes a) = \langle \pi(\phi'(a))p(d)\xi, \xi \rangle$. In fact, this is just the composition

$$(2.3) \quad C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A} \xrightarrow{id \otimes \phi'} C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{B} \xrightarrow{p \times \pi} \mathbb{B}(\mathcal{H}) \xrightarrow{\langle (-)\xi, \xi \rangle} \mathbb{C}.$$

Let \mathcal{C}_ρ be the weak- $*$ closure of $\{\rho_{\psi'} : \psi' \in K(\mathcal{S})\}$. Since $K(\mathcal{S})$ is a closed operator convex cone and ξ is cyclic for the image of π , one easily checks (as done in the proof of [KR02, Lemma 7.18]) that \mathcal{C}_ρ is a cone such that $\rho'(z^*(-)z)$ is in \mathcal{C}_ρ for all $z \in C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}$ and all $\rho' \in \mathcal{C}_\rho$.

Let $\mathfrak{I} = \{z \in C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A} : \rho'(z^*z) = 0, \text{ for all } \rho' \in \mathcal{C}_\rho\}$. By [KR02, Lemma 7.17 (ii)] \mathfrak{I} is a two-sided, closed ideal in $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}$ such that any positive linear functional vanishing on \mathfrak{I} is in \mathcal{C}_ρ . Thus if ρ_ϕ vanished on \mathfrak{I} , then $\rho_\phi \in \mathcal{C}_\rho$ which implies that there would be a $\psi \in K(\mathcal{S})$ such that $|\rho_\phi(a_i \otimes d_i) - \rho_\psi(a_i \otimes d_i)| < \epsilon$ for $i = 1, \dots, n$. However, this would imply that

$$f_i(\phi(a_i)) = \langle \pi(\phi(a_i))p(d_i)\xi, \xi \rangle = \rho_\phi(a_i \otimes d_i) \approx_\epsilon \rho_\psi(a_i \otimes d_i) = f_i(\psi(a_i)),$$

a contradiction. Thus there is a $z \in \mathfrak{I}$ such that $\rho_\phi(z^*z) > 0$. Since \mathfrak{I} is a two-sided, closed ideal containing the generators of $\mathfrak{I}(z^*z)$, it follows that $\mathfrak{I}(z^*z) \subset \mathfrak{I}$, and thus $(id \otimes \phi)(z^*z) \notin \mathfrak{I}(z^*z)$. \square

Corollary 2.19. *Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone and $\phi \in CP(\mathfrak{A}, \mathfrak{B})$. Then $\phi \in \mathcal{C}$ if and only if for every positive $c \in C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}$, the element $(id \otimes \phi)(c)$ is in the two-sided, closed ideal $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{B}$ generated by*

$$\{(id \otimes \psi)(c) : \psi \in \mathcal{C}\}.$$

Proof. This follows from Theorem 2.18 since $(id \otimes \psi)((1 \otimes a^*)c(1 \otimes a)) = (id \otimes \psi(a^*(-)a))(c)$ and $\psi(a^*(-)a)$ is in \mathcal{C} , for all $\psi \in \mathcal{C}$ and $a \in \mathfrak{A}$. \square

Using that two-sided, closed ideals are hereditary C^* -subalgebras, we immediately obtain the following corollary. Although the corollary will not be used in this paper, it somehow illustrates how large a closed operator convex cone necessarily is, even though these are often generated by only a single c.p. map (see e.g. Corollary 3.18).

Corollary 2.20. *Any closed operator convex cone \mathcal{C} is hereditary, i.e. if ϕ, ψ are c.p. maps such that $\phi + \psi \in \mathcal{C}$, then $\phi, \psi \in \mathcal{C}$.*

Proof of Proposition 2.17. Since \mathfrak{D} is nuclear we have

$$C^*(\mathbb{F}_\infty) \otimes_{\max} (\mathfrak{B} \otimes \mathfrak{D}) = (C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{B}) \otimes \mathfrak{D},$$

thus we will simply write $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{B} \otimes \mathfrak{D}$. Since \mathfrak{D} is simple and nuclear, there is a lattice isomorphism between the lattice of two-sided, closed ideals in $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{B}$ and in $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{B} \otimes \mathfrak{D}$ given by $\mathfrak{J} \mapsto \mathfrak{J} \otimes \mathfrak{D}$.

Let $\mathfrak{C} = C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{B}$. For a subset $\mathcal{S} \subset CP(\mathfrak{A}, \mathfrak{B})$ and $c \in C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}$, let $I_{\mathcal{S}}(c)$ be the two-sided, closed ideal in \mathfrak{C} generated by

$$\{(id \otimes \psi)((1 \otimes a^*)c(1 \otimes a)) : \psi \in \mathcal{S}, a \in \mathfrak{A}\}.$$

By Theorem 2.18 a c.p. map $\phi \in CP(\mathfrak{A}, \mathfrak{B})$ is in $K(\mathcal{S})$ if and only if $(id \otimes \phi)(c) \in I_{\mathcal{S}}(c)$ for all positive $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$.

Similarly, for a subset $\mathcal{S}_0 \subset CP(\mathfrak{A}, \mathfrak{B} \otimes \mathfrak{D})$ and $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$, let $J_{\mathcal{S}_0}(c)$ be the two-sided, closed ideal in \mathfrak{C} (the existence of which follows from the lattice isomorphism above), such that $J_{\mathcal{S}_0}(c) \otimes \mathfrak{D}$ is the two-sided, closed ideal generated by

$$\{(id \otimes \psi)((1 \otimes a^*)c(1 \otimes a)) : \psi \in \mathcal{S}_0, a \in \mathfrak{A}\}.$$

Again by Theorem 2.18 a c.p. map $\psi \in CP(\mathfrak{A}, \mathfrak{B} \otimes \mathfrak{D})$ is in $K(\mathcal{S}_0)$ if and only if $(id \otimes \psi)(c) \in J_{\mathcal{S}_0}(c) \otimes \mathfrak{D}$ for all positive $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$.

Let F be the map which takes a closed operator convex cone \mathcal{K} in $CP(\mathfrak{A}, \mathfrak{B} \otimes \mathfrak{D})$ to

$$K(\{(id_{\mathfrak{B}} \otimes \eta) \circ \psi : \psi \in \mathcal{K}, \eta \in S(\mathfrak{D})\}),$$

where $S(\mathfrak{D})$ is the space of states on \mathfrak{D} . Note that this is the closed operator convex cone *generated* by the set which, in the statement of the proposition, is claimed to be a closed operator convex cone.

We will show that $F(\mathcal{C} \otimes \mathfrak{D}) = \mathcal{C}$ and $F(\mathcal{K}) \otimes \mathfrak{D} = \mathcal{K}$. We will use the following facts which we state without proof.

Fact 1: Let $\mathfrak{S} \subset \mathfrak{C}$ be a set of positive elements and $\mathfrak{J}(\mathfrak{S})$ be the two-sided, closed ideal in \mathfrak{C} generated by \mathfrak{S} . Then for any positive non-zero $d \in \mathfrak{D}$ (which is full in \mathfrak{D} since \mathfrak{D} is simple), $\mathfrak{J}(\mathfrak{S}) \otimes \mathfrak{D}$ is the two-sided, closed ideal generated by the set

$$\{x \otimes d : x \in \mathfrak{S}\}.$$

Fact 2: (Cf. [Bla06, Corollary IV.3.4.2]) Similarly, suppose $\mathfrak{S}_0 \subset \mathfrak{C} \otimes \mathfrak{D}$. Let $\mathfrak{J}(\mathfrak{S}_0)$ be the two-sided, closed ideal in \mathfrak{C} such that $\mathfrak{J}(\mathfrak{S}_0) \otimes \mathfrak{D}$ is the two-sided, closed ideal in $\mathfrak{C} \otimes \mathfrak{D}$ generated by \mathfrak{S}_0 (the existence of $\mathfrak{J}(\mathfrak{S}_0)$ follows from what we noted above). Then $\mathfrak{J}(\mathfrak{S}_0)$ is the two-sided, closed ideal generated by the set

$$\{(id_{\mathfrak{C}} \otimes \eta)(y) : y \in \mathfrak{S}_0, \eta \in S(\mathfrak{D})\}.$$

If \mathfrak{D} is separable, it suffices to take the η above only in a countable dense subset of $S(\mathfrak{D})$.

From Fact 1 it follows that if $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ is a closed operator convex cone, then $I_{\mathcal{C}}(c) = J_{\mathcal{C} \otimes \mathfrak{D}}(c)$ for any positive $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$. To see this, let $\mathcal{C}_d := \{\phi \otimes d : \phi \in \mathcal{C}\}$ and note that by definition $\mathcal{C} \otimes \mathfrak{D} = K(\mathcal{C}_d)$. Thus from Fact 1 we get that

$$I_{\mathcal{C}}(c) \otimes \mathfrak{D} = ideal(\{(id \otimes \phi \otimes d)((1 \otimes a)^*c(1 \otimes a)) : \phi \in \mathcal{C}, a \in \mathfrak{A}\}) = J_{\mathcal{C}_d}(c) \otimes \mathfrak{D} = J_{\mathcal{C} \otimes \mathfrak{D}}(c) \otimes \mathfrak{D},$$

where $ideal(\mathfrak{S}_0)$ means the two-sided, closed ideal generated by \mathfrak{S}_0 .

Similarly, from Fact 2 it follows that if $\mathcal{K} \subset CP(\mathfrak{A}, \mathfrak{B} \otimes \mathfrak{D})$ is a closed operator convex cone, then $I_{F(\mathcal{K})}(c) = J_{\mathcal{K}}(c)$ for any positive $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$. Thus

$$I_{F(\mathcal{C} \otimes \mathfrak{D})}(c) = J_{\mathcal{C} \otimes \mathfrak{D}}(c) = I_{\mathcal{C}}(c)$$

and

$$J_{F(\mathcal{K}) \otimes \mathfrak{D}}(c) = I_{F(\mathcal{K})}(c) = J_{\mathcal{K}}(c),$$

for all positive $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$. Thus it follows that $\mathcal{C} = F(\mathcal{C} \otimes \mathfrak{D})$ and $\mathcal{K} = F(\mathcal{K}) \otimes \mathfrak{D}$.

It remains to show that $\mathcal{S} := \{(id_{\mathfrak{B}} \otimes \eta) \circ \psi : \psi \in \mathcal{K}, \eta \in S(\mathfrak{D})\}$ is a closed operator convex cone for any closed operator convex cone \mathcal{K} , and that \mathcal{C} is countably generated if and only if $\mathcal{C} \otimes \mathfrak{D}$ is countably generated.

It follows from what we have already proven that \mathcal{K} above is of the form $\mathcal{C} \otimes \mathfrak{D}$. Thus for any $\phi \in \mathcal{C}$, $\phi \otimes d \in \mathcal{K}$. Choosing a state η on \mathfrak{D} such that $\eta(d) > 0$, it follows that

$$\phi = (id \otimes \eta) \circ ((\eta(d)^{-1}\phi) \otimes d) \in \mathcal{S}$$

and thus $\mathcal{C} \subset \mathcal{S}$. Since $K(\mathcal{S}) = \mathcal{C}$ it follows that $\mathcal{S} = \mathcal{C}$.

It follows from Fact 1 that $\mathcal{C} \otimes \mathfrak{D}$ is countably generated if \mathcal{C} is countably generated. To see this, let $\mathcal{S} \subset \mathcal{C}$ be a countable subset generating \mathcal{C} . Then $I_{\mathcal{S}}(c) = I_{\mathcal{C}}(c) = J_{\mathcal{C} \otimes \mathfrak{D}}(c)$ for all positive $c \in C^*(\mathbb{F}_{\infty}) \otimes_{\max} \mathfrak{A}$. Let $\mathcal{S}_d = \{\phi \otimes d : \phi \in \mathcal{S}\}$. It follows from Fact 1 that

$$J_{\mathcal{C} \otimes \mathfrak{D}}(c) \otimes \mathfrak{D} = I_{\mathcal{S}}(c) \otimes \mathfrak{D} = ideal(\{(id \otimes \phi \otimes d)((1 \otimes a)^* c (1 \otimes a)) : \phi \in \mathcal{S}, a \in \mathfrak{A}\}) = J_{\mathcal{S}_d}(c) \otimes \mathfrak{D}$$

which implies that $\mathcal{C} \otimes \mathfrak{D}$ is generated by the countable set \mathcal{S}_d .

A similar argument implies, that if \mathfrak{D} is separable then $F(\mathcal{K})$ is countably generated if \mathcal{K} is countably generated. \square

Notation 2.21. Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. We let $\mathcal{C}^s \subset CP(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K})$ denote the closed operator convex cone generated by \mathcal{C} by Proposition 2.17.

In this section we will only apply the following lemma when p is an isomorphism. However, the more general statement will be applied when proving a Choi–Effros type theorem, Corollary 5.31.

Lemma 2.22. *Let $\mathfrak{A}, \mathfrak{B}$ and \mathfrak{C} be C^* -algebras with \mathfrak{A} separable, and let $p: \mathfrak{B} \rightarrow \mathfrak{C}$ be a surjective $*$ -homomorphism. Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Then*

$$p(\mathcal{C}) := \{p \circ \phi : \phi \in \mathcal{C}\}$$

is a closed operator convex cone. Also, the set

$$\{p \circ \phi : \phi \in \mathcal{C}, \|\phi\| \leq 1\}$$

is point-norm closed.

Proof. Clearly $p(\mathcal{C})$ is an operator convex cone. The proof that the two sets above are point-norm closed is essentially the exact same proof as [Arv77, Theorem 6] (that the set of liftable (contractive) c.p. maps is closed). \square

Using Theorem 2.18 the following is obvious.

Corollary 2.23. *Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone and suppose that $\mathfrak{B} = \mathfrak{B}_0 \otimes \mathbb{K}$. If $\Psi': \mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K}$ is an isomorphism and $\Psi = id_{\mathfrak{B}_0} \otimes \Psi': \mathfrak{B} \otimes \mathbb{K} \rightarrow \mathfrak{B}$ is the induced isomorphism, then $\Psi(\mathcal{C}^s) = \mathcal{C}$.*

Later in the paper, we will also need the following lemma, which looks similar to Lemma 2.22 but is very different in nature.

Lemma 2.24. *Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone, and let $p: \mathfrak{C} \rightarrow \mathfrak{A}$ be a surjective $*$ -homomorphism. Then*

$$\mathcal{C}_0 := \{\phi \circ p : \phi \in \mathcal{C}\}$$

is a closed operator convex cone. Moreover, if $\mathcal{S} \subset \mathcal{C}$ generates \mathcal{C} , then

$$\mathcal{S}_0 := \{\phi \circ p : \phi \in \mathcal{S}\}$$

generates \mathcal{C}_0 .

Proof. Clearly \mathcal{C}_0 is an operator convex cone. Let $(\phi_\lambda \circ p)$ be a net in \mathcal{C}_0 , with $\phi_\lambda \in \mathcal{C}$, which converges point-norm to a c.p. map $\psi: \mathfrak{A} \rightarrow \mathfrak{B}$. Since every $\phi_\lambda \circ p$ vanishes on $\ker p$, so does ψ and thus $\psi = \phi \circ p$ for some c.p. map $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$. Clearly ϕ_λ converges point-norm to ϕ so $\phi \in \mathcal{C}$. It follows that $\psi \in \mathcal{C}_0$, so \mathcal{C}_0 is point-norm closed.

Suppose \mathcal{S} generates \mathcal{C} . Let $\psi \in \mathcal{C}_0$ and write $\psi = \phi \circ p$ with $\phi \in \mathcal{C}$. By Remark 2.6, ϕ may be approximated point-norm by sums of maps of the form

$$a \mapsto \sum_{i,j=1}^n b_i^* \phi'(a_i^* a a_j) b_j = \sum_{i,j=1}^n b_i^* \phi'(p(c_i)^* a p(c_j)) b_j$$

with $\phi' \in \mathcal{S}$, and where $p(c_i) = a_i$ for each i . It easily follows that $\phi \circ p$ may be approximated point-norm by sums of maps of the form

$$c \mapsto \sum_{i,j=1}^n b_i^* \phi'(p(c_i)^* p(c) p(c_j)) b_j = \sum_{i,j=1}^n b_i^* \psi'(c_i^* c c_j) b_j$$

where $\psi' = \phi' \circ p \in \mathcal{S}_0$. Hence \mathcal{S}_0 generates \mathcal{C}_0 . \square

Recall that if $\mathcal{H}_{\mathfrak{B}} = \bigoplus_{\mathbb{N}} \mathfrak{B}$ is the canonical Hilbert \mathfrak{B} -module, then there is a canonical isomorphism $\eta: \mathbb{B}(\mathcal{H}_{\mathfrak{B}}) \rightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$ for which the restriction $\mathbb{K}(\mathcal{H}_{\mathfrak{B}}) \rightarrow \mathfrak{B} \otimes \mathbb{K}$ is also an isomorphism, the restricted isomorphism given by $\theta_{(b)_n, (c)_m} \mapsto bc^* \otimes e_{nm}$. Here $(b)_n \in \mathcal{H}_{\mathfrak{B}}$ denotes the element which is b on the n 'th coordinate and zero everywhere else.

Corollary 2.25. *With the notation as above, we have*

$$\mathcal{C}^{\mathcal{S}} = \eta(\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}).$$

Moreover, \mathcal{C} is countably generated if and only if $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$ is countably generated.

Proof. This follows easily from Proposition 2.17 and Theorem 2.18. \square

Thus combining the Kasparov–Stinespring Theorem for Hilbert modules with Corollaries 2.23 and 2.25 we get the following multiplier algebra version.

Corollary 2.26 (Kasparov–Stinespring Theorem for multiplier algebras). *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras with \mathfrak{A} separable and \mathfrak{B} σ -unital and stable, let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone, and let $\phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ be a contractive completely positive map weakly in \mathcal{C} . Then there is a $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ weakly in \mathcal{C} and isometries $V, W \in \mathcal{M}(\mathfrak{B})$ with $VV^* + WW^* = 1$, such that $V^* \Phi(-) V = \phi$.*

If, in addition, \mathfrak{A} is unital \mathcal{C} is non-degenerate, then there is a unital $$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ weakly in \mathcal{C} and an element $V \in \mathcal{M}(\mathfrak{B})$ such that $V^* \Phi(-) V = \phi$. If ϕ is unital, then such a V exists, for which there is another isometry W with $VV^* + WW^* = 1$.*

We also get another corollary which will be applied by the first named author in [Gab15].

Corollary 2.27. *Let \mathfrak{A} be a separable, unital C^* -algebra, \mathfrak{B} be a σ -unital C^* -algebra, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Then there exists a unital c.p. map $\phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ weakly in \mathcal{C} if and only if \mathcal{C} is non-degenerate.*

Proof. If ϕ is unital and weakly in \mathcal{C} , and $h \in \mathfrak{B}$ is a strictly positive element, then $h^{1/2} \phi(-) h^{1/2}$ does not factor through any proper, two-sided, closed ideal in \mathfrak{J} . Thus \mathcal{C} is non-degenerate.

Conversely, it easily follows from Proposition 2.17 that \mathcal{C}^s is also non-degenerate. By Corollary 2.14 and Corollary 2.25 there is a unital $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$ weakly in \mathcal{C}^s . Let $\psi = (1 \otimes e_{11})\Phi(-)(1 \otimes e_{11}): \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B}) \otimes e_{11} \subset \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$. Regarding this as a c.p. map $\mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ it follows by Proposition 2.17 that this c.p. map is weakly in \mathcal{C} . Since Φ is unital it also follows that ψ is unital. \square

3. ABSORBING REPRESENTATIONS

We find necessary and sufficient conditions for when there exist representations weakly in \mathcal{C} which absorb any representation weakly in \mathcal{C} . We obtain this result in a unital and a non-unital version.

Definition 3.1. Let \mathfrak{A} and \mathfrak{B} be C^* -algebras with \mathfrak{A} separable and \mathfrak{B} σ -unital, and let $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ and $\Psi: \mathfrak{A} \rightarrow \mathbb{B}(F)$ be representations with E and F countably generated Hilbert \mathfrak{B} -modules. We say that Φ is *approximately unitarily equivalent* to Ψ , written $\Phi \sim_{ap} \Psi$, if there is a sequence of unitaries (U_n) in $\mathbb{B}(E, F)$ such that

- (i) $\Phi(a) - U_n^* \Psi(a) U_n \in \mathbb{K}(E)$, for all $a \in \mathfrak{A}$ and $n \in \mathbb{N}$,
- (ii) $\lim_{n \rightarrow \infty} \|\Phi(a) - U_n^* \Psi(a) U_n\| = 0$, for all $a \in \mathfrak{A}$.

We say that Φ is *asymptotically unitarily equivalent*¹ to Ψ , written $\Phi \sim_{as} \Psi$, if there is a norm continuous path $U: [1, \infty) \rightarrow \mathbb{B}(E, F)$ of unitaries such that

- (i') $\Phi(a) - U_t^* \Psi(a) U_t \in \mathbb{K}(E)$, for all $a \in \mathfrak{A}$ and $t \in [1, \infty)$,
- (ii') $\lim_{t \rightarrow \infty} \|\Phi(a) - U_t^* \Psi(a) U_t\| = 0$, for all $a \in \mathfrak{A}$.

We will also need the following related notions.

Definition 3.2. Let \mathfrak{A} and \mathfrak{B} be C^* -algebras with \mathfrak{A} separable and \mathfrak{B} σ -unital, let $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ be a representation and $\phi: \mathfrak{A} \rightarrow \mathbb{B}(F)$ be a c.p. map, where E and F are countably generated Hilbert \mathfrak{B} -modules. We say that Φ *approximately dominates* ϕ if there is a bounded sequence (v_n) in $\mathbb{B}(F, E)$ such that

- (i) $v_n^* \Phi(a) v_n - \phi(a) \in \mathbb{K}(F)$ for all $a \in \mathfrak{A}$ and all $n \in \mathbb{N}$,
- (ii) $\lim_{n \rightarrow \infty} \|v_n^* \Phi(a) v_n - \phi(a)\| = 0$ for all $a \in \mathfrak{A}$.

Furthermore, we say that Φ *strongly approximately dominates* ϕ if we may pick a sequence (v_n) satisfying (i) and (ii) above, and also

- (iii) $\lim_{n \rightarrow \infty} \|v_n^* T v_n\| = 0$ for all $T \in \mathbb{K}(E)$.

Moreover, if we may find a norm-continuous bounded path $(v_t)_{t \in [1, \infty)}$ satisfying the obvious analogues of (i), (ii) (and (iii)) above, then we say that Φ *(strongly) asymptotically dominates* ϕ .

Remark 3.3. If \mathfrak{A} is unital and Φ and ϕ are both unital in the above definition, then we may always pick the bounded sequence (v_n) (or family (v_t)) to consist of isometries. This follows easily since unitality implies that $v_n^* v_n - 1$ is compact and small in norm for large n . Thus for large n , $w_n := v_n(v_n^* v_n)^{-1/2}$ is an isometry identical to v_n modulo compacts, and w_n makes (i), (ii) (and (iii)) hold for Φ and ϕ if v_n does.

Theorem 3.4. Let \mathfrak{A} and \mathfrak{B} be C^* -algebras with \mathfrak{A} separable and unital, and \mathfrak{B} σ -unital. Let $\Phi, \Psi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ be unital representations and let $\Psi_{\infty}: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}}^{\infty})$ be the infinite repeat of Ψ . The following are equivalent.

¹Sometimes called *unitarily homotopic*.

- (i) Φ strongly approximately dominates $T^*\Psi(-)T$ for every $T \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$.
- (ii) Φ strongly approximately dominates Ψ ,
- (iii) Φ strongly asymptotically dominates Ψ ,
- (iv) There is a unitary $U \in \mathbb{B}(\mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}}^\infty, \mathcal{H}_{\mathfrak{B}})$ such that

$$U^*\Phi(a)U - \Phi(a) \oplus \Psi_\infty(a) \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}}^\infty) \text{ for all } a \in \mathfrak{A},$$

- (v) $\Phi \oplus \Psi_\infty \sim_{ap} \Phi$,
- (vi) $\Phi \oplus \Psi_\infty \sim_{as} \Phi$.

Proof. We will prove $(v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (v)$ and $(iii) \Rightarrow (vi) \Rightarrow (v)$.

$(v) \Rightarrow (iv)$: Obvious.

$(iv) \Rightarrow (iii)$: Let U be as in (iv). Let $V_n \in \mathbb{B}(\mathcal{H}_{\mathfrak{B}}, \mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}}^\infty)$ be the isometry given as the inclusion into the n 'th coordinate of $\mathcal{H}_{\mathfrak{B}}^\infty$. Let $V_t := (n+1-t)^{1/2}V_n + (t-n)^{1/2}V_{n+1}$ for $t \in [n, n+1]$. Then $(V_t)_{t \in [1, \infty)}$ is a continuous family of isometries such that $V_t^*TV_t \rightarrow 0$ for all $T \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}}^\infty)$. Let $W_t := UV_t$. Since $V_t^*(\Phi(a) \oplus \Psi_\infty(a))V_t = \Psi(a)$ for all $a \in \mathfrak{A}$, it follows $W_t^*\Phi(a)W_t - \Psi(a)$ is in $\mathbb{K}(\mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}}^\infty)$ and tends to 0 as $t \rightarrow \infty$, for all $a \in \mathfrak{A}$.

$(iii) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (i)$: If W_n implements a strong approximate domination of Ψ , and $T \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ is given, then $V_n = TW_n$ implements a strong approximate domination of $T^*\Psi(-)T$.

$(i) \Rightarrow (v)$: This follows from [DE02, Theorem 2.13].

$(iii) \Rightarrow (vi)$: By $(iii) \Leftrightarrow (iv)$ as proven above, and by identifying Ψ_∞ and $(\Psi_\infty)_\infty$ (which are clearly unitarily equivalent), it follows that Φ asymptotically dominates Ψ_∞ . Let $V_t \in \mathbb{B}(\mathcal{H}_{\mathfrak{B}}^\infty, \mathcal{H}_{\mathfrak{B}})$ be a bounded continuous family of elements such that $V_t^*\Phi(a)V_t - \Psi_\infty(a)$ is compact and tends to zero for every a in \mathfrak{A} . By a standard trick of Arveson [Arv77, Cf. proof of Corollary 1] it follows that $V_t\Psi_\infty(a) - \Phi(a)V_t$ is compact and tends to zero for all a in \mathfrak{A} . By, once again, identifying Ψ_∞ and $(\Psi_\infty)_\infty$, it follows from [DE02, Lemma 2.16] that $\Phi \oplus \Psi_\infty \sim_{as} \Phi$.

$(vi) \Rightarrow (v)$: Obvious. \square

Definition 3.5. Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. A (unital) representation $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$, is called *(unitally) \mathcal{C} -absorbing*, if for any (unital) representation $\Psi: \mathfrak{A} \rightarrow \mathbb{B}(F)$ weakly in \mathcal{C} , we have that $\Phi \oplus \Psi \sim_{as} \Phi$. Here we implicitly assume that E and F are countably generated Hilbert \mathfrak{B} -modules.

Remark 3.6. Suppose that $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ is a \mathcal{C} -absorbing representation. Since Φ absorbs the zero representation $\mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ it follows that $E \cong E \oplus \mathcal{H}_{\mathfrak{B}} \cong \mathcal{H}_{\mathfrak{B}}$ by Kasparov's stabilisation theorem.

Also, if $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ is a unitally \mathcal{C} -absorbing representation and \mathcal{C} is non-degenerate, then $E \cong \mathcal{H}_{\mathfrak{B}}$ since it absorbs a unital representation $\mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ weakly in \mathcal{C} , which exists by Corollary 2.14.

Thus it is essentially no loss of generality only to consider the case when $E = \mathcal{H}_{\mathfrak{B}}$, which is also the case that corresponds to the multiplier algebra picture.

Remark 3.7. Note that any two (unital) \mathcal{C} -absorbing representations Φ and Φ' which are weakly in \mathcal{C} , are asymptotically unitarily equivalent since $\Phi \sim_{as} \Phi \oplus \Phi' \sim_{as} \Phi'$.

Remark 3.8. By Theorem 3.4, a unital representation Φ is unitally \mathcal{C} -absorbing if and only if $\Phi \sim_{ap} \Phi \oplus \Psi$ for any unital representation Ψ weakly in \mathcal{C} .

The same is true in the non-unital case, which follows easily from Proposition 3.10 below.

The following lemma shows, that given a closed operator convex cone in $CP(\mathfrak{A}, \mathfrak{B})$ there is an induced closed operator convex cone in $CP(\mathfrak{A}^\dagger, \mathfrak{B})$. Recall that \mathfrak{A}^\dagger denotes the forced unitisation of \mathfrak{A} , i.e. we add a unit to \mathfrak{A} regardless if \mathfrak{A} is unital or not.

Lemma 3.9. *Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone and define*

$$\mathcal{C}^\dagger := \{\phi \in CP(\mathfrak{A}^\dagger, \mathfrak{B}) : \phi|_{\mathfrak{A}} \in \mathcal{C}\}.$$

Then \mathcal{C}^\dagger is a non-degenerate, closed operator convex cone.

Proof. Clearly \mathcal{C}^\dagger is a cone. Let $\phi \in \mathcal{C}^\dagger$. Clearly $b^*\phi(-)b$ is in \mathcal{C}^\dagger for every $b \in \mathfrak{B}$. Let $a_1, \dots, a_n \in \mathfrak{A}^\dagger$ and $b_1, \dots, b_n \in \mathfrak{B}$. We should show that if $\psi: \mathfrak{A}^\dagger \rightarrow \mathfrak{B}$ is given by

$$\psi(a) = \sum_{i,j}^n b_i^* \phi(a_i^* a a_j) b_j,$$

then $\psi|_{\mathfrak{A}} \in \mathcal{C}$. If \mathfrak{A} is non-unital, the $\mathfrak{A}^\dagger \subset \mathcal{M}(\mathfrak{A})$ and the result follows from Remark 2.5. If \mathfrak{A} is unital, then $\mathfrak{A}^\dagger = \mathfrak{A} \oplus \mathbb{C}$, and $a_i = a'_i \oplus \lambda_i$. Thus when we restrict ψ to \mathfrak{A} , we may replace the a_i with $a'_i \in \mathfrak{A}$, and thus ψ is in \mathcal{C}^\dagger . It follows that \mathcal{C}^\dagger is an operator convex cone. If (ϕ_n) is a sequence in \mathcal{C}^\dagger which converges point-norm to a c.p. map $\phi: \mathfrak{A}^\dagger \rightarrow \mathfrak{B}$, then $\phi_n|_{\mathfrak{A}} \rightarrow \phi|_{\mathfrak{A}}$ point-norm and thus $\phi|_{\mathfrak{A}} \in \mathcal{C}$. Hence \mathcal{C} is point-norm closed. To see that \mathcal{C}^\dagger is non-degenerate, let for any positive $b \in \mathfrak{B}$, $\rho_b: \mathbb{C} \rightarrow \mathfrak{B}$ be the c.p. map given by $\rho_b(1) = b$. Clearly the composition

$$\mathfrak{A}^\dagger \rightarrow \mathfrak{A}^\dagger / \mathfrak{A} \cong \mathbb{C} \xrightarrow{\rho_b} \mathfrak{B}$$

is in \mathcal{C}^\dagger for every positive $b \in \mathfrak{B}$, and thus \mathcal{C}^\dagger is non-degenerate. \square

The following lemma tells us, that in our quest to find absorbing representations, we may often restrict to the unital case and vice versa.

Proposition 3.10. *Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Then $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ is a \mathcal{C} -absorbing representation if and only if $\Phi^\dagger: \mathfrak{A}^\dagger \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ is a unittally \mathcal{C}^\dagger -absorbing representation.*

Moreover, if \mathfrak{A} is unital, \mathcal{C} is non-degenerate, and $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ is a unital representation, then Φ is unittally \mathcal{C} -absorbing if and only if $\Phi \oplus 0: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}})$ is \mathcal{C} -absorbing.

Proof. Suppose Φ^\dagger is \mathcal{C}^\dagger -absorbing. Then clearly Φ is \mathcal{C} -absorbing since every representation weakly in \mathcal{C} extends to a unital representation weakly in \mathcal{C}^\dagger and the induced asymptotic unitary equivalences in the unital case clearly hold in the non-unital case, since we just restrict to \mathfrak{A} .

Now suppose that Φ is \mathcal{C} -absorbing. Let $\Psi: \mathfrak{A}^\dagger \rightarrow \mathbb{B}(E)$ be a unital representation weakly in \mathcal{C}^\dagger . Since $\Psi|_{\mathfrak{A}}$ is weakly in \mathcal{C} we may find a path of unitaries (U_t) in $\mathbb{B}(\mathcal{H}_{\mathfrak{B}}, \mathcal{H}_{\mathfrak{B}} \oplus E)$ such that conditions (i') and (ii') of Definition 3.1 are satisfied. Then

$$U_t^*(\Phi^\dagger \oplus \Psi)(a + \lambda 1)U_t - \Phi^\dagger(a + \lambda 1) = U_t^*(\Phi \oplus \Psi|_{\mathfrak{A}})(a)U_t - \Phi(a)$$

and thus Φ^\dagger clearly absorbs Ψ .

Now suppose that \mathfrak{A} and Φ are unital, and that \mathcal{C} is non-degenerate. Suppose that Φ is unittally \mathcal{C} -absorbing and let $\Psi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ be a representation weakly in \mathcal{C} . Letting $P = \Psi(1_{\mathfrak{A}})$ we get that $E = PE \oplus P^\perp E$, and clearly $\Psi = \Psi_1 \oplus 0: \mathfrak{A} \rightarrow \mathbb{B}(PE \oplus P^\perp E)$ for a unital representation Ψ_1 weakly in \mathcal{C} . Letting (U_t) be a continuous path of unitaries in

$\mathbb{B}(\mathcal{H}_{\mathfrak{B}}, \mathcal{H}_{\mathfrak{B}} \oplus PE)$ implementing the asymptotic unitary equivalence of $\Phi \oplus \Psi_1$ and Φ , we clearly have that

$$U_t \oplus 1 \in \mathbb{B}(\mathcal{H}_{\mathfrak{B}} \oplus (P^\perp E \oplus \mathcal{H}_{\mathfrak{B}}), (\mathcal{H}_{\mathfrak{B}} \oplus PE) \oplus (P^\perp E \oplus \mathcal{H}_{\mathfrak{B}}))$$

implements an asymptotic unitary equivalence of $\Phi \oplus \Psi \oplus 0_{\mathcal{H}_{\mathfrak{B}}}$ and $\Phi \oplus 0_{P^\perp E \oplus \mathcal{H}_{\mathfrak{B}}}$. The result now follows since $\Phi \oplus \Psi \oplus 0_{\mathcal{H}_{\mathfrak{B}}}$ is unitarily equivalent to $\Phi \oplus 0_{\mathcal{H}_{\mathfrak{B}}} \oplus \Psi$ and $\Phi \oplus 0_{P^\perp E \oplus \mathcal{H}_{\mathfrak{B}}}$ is unitarily equivalent to $\Phi \oplus 0_{\mathcal{H}_{\mathfrak{B}}}$ by Kasparov's stabilisation theorem.

Suppose $\Phi \oplus 0$ is \mathcal{C} -absorbing and let $\Psi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ be a unital representation weakly in \mathcal{C} . We first prove the case where $E \cong \mathcal{H}_{\mathfrak{B}}$. Consider the infinite repeat $\Psi_\infty: \mathfrak{A} \rightarrow \mathbb{B}(F)$ where $F = E^\infty$. We may find a continuous path of unitaries in $\mathbb{B}(\mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}}, \mathcal{H}_{\mathfrak{B}} \oplus F \oplus \mathcal{H}_{\mathfrak{B}})$ such that

$$U_t^*(\Phi(a) \oplus \Psi_\infty(a) \oplus 0)U_t - \Phi(a) \oplus 0$$

is compact and tends to zero. Let $S \in \mathbb{B}(\mathcal{H}_{\mathfrak{B}} \oplus F, (\mathcal{H}_{\mathfrak{B}} \oplus F) \oplus \mathcal{H}_{\mathfrak{B}})$ and $T \in \mathbb{B}(\mathcal{H}_{\mathfrak{B}}, \mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}})$ be the isometries which are the embedding into the first summands. By letting $V_t = S^*U_tT$ we have a contractive continuous family such that

$$V_t^*(\Phi(a) \oplus \Psi_\infty(a))V_t - \Phi(a) = T^*(U_t^*(\Phi(a) \oplus \Psi_\infty(a) \oplus 0)U_t - \Phi(a) \oplus 0)T$$

is compact and tends to zero. It easily follows that Φ strongly asymptotically dominates Ψ and by Theorem 3.4 Φ absorbs Ψ .

Now suppose that E is not isomorphic to $\mathcal{H}_{\mathfrak{B}}$. By Corollary 2.14 there is a unital representation $\Psi_1: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ weakly in \mathcal{C} . Since $\mathcal{H}_{\mathfrak{B}} \oplus E \cong \mathcal{H}_{\mathfrak{B}}$ by Kasparov's stabilisation theorem, it follows from what we proved above that Φ absorbs both Ψ_1 and $\Psi_1 \oplus \Psi$. Hence

$$\Phi \oplus \Psi \sim_{as} \Phi \oplus \Psi_1 \oplus \Psi \sim_{as} \Phi. \quad \square$$

Lemma 3.11. *Any \mathcal{C} -absorbing representation strongly asymptotically dominates every c.p. map weakly in \mathcal{C} . Also, if \mathcal{C} is non-degenerate, then any unital \mathcal{C} -absorbing representation strongly asymptotically dominates every c.p. map weakly in \mathcal{C} .*

Proof. We do the unital case first. Suppose $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(E)$ is a unital \mathcal{C} -absorbing and $\psi: \mathfrak{A} \rightarrow \mathbb{B}(F)$ is a c.p. map weakly in \mathcal{C} . By rescaling ψ if necessary, we may assume that ψ is contractive. Let (Ψ, V) be a unital Kasparov–Stinespring dilation of ψ weakly in \mathcal{C} by Theorem 2.13. By Theorem 3.4 Φ strongly asymptotically dominates Ψ , so let W_t be a bounded family implementing this strong asymptotic domination. Then W_tV implements a strong asymptotic domination of ψ .

Now consider the non-unital case. Let Φ be \mathcal{C} -absorbing and ψ be weakly in \mathcal{C} . By Proposition 3.10 Φ^\dagger is unital \mathcal{C}^\dagger -absorbing. By scaling ψ we may assume that ψ is contractive, and thus we may extend ψ to a unital c.p. map ψ^\dagger . Clearly ψ^\dagger is weakly in \mathcal{C}^\dagger , and thus Φ^\dagger strongly asymptotically dominates ψ^\dagger . By restricting to $\mathfrak{A} \subset \mathfrak{A}^\dagger$, it follows that Φ strongly asymptotically dominates ψ . \square

We may now prove an important theorem. In some sense, the following theorem has been a main ingredient in many absorption results to date. This will be explained in more detail in Remark 3.13 below.

Theorem 3.12. *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras with \mathfrak{A} separable and unital, and \mathfrak{B} σ -unital, and let $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ be a unital representation. Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a non-degenerate, closed operator convex cone and suppose that the subset $\mathcal{S} \subset CP(\mathfrak{A}, \mathbb{K}(\mathcal{H}_{\mathfrak{B}}))$ generates $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$.*

If Φ strongly approximately dominates every map in \mathcal{S} , then Φ strongly approximately dominates any completely positive map $\mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ weakly in \mathcal{C} .

In particular, Φ is unittally \mathcal{C} -absorbing if and only if Φ strongly approximately dominates every map in \mathcal{S} .

Proof. This is proven using methods from [EK01]. First, we will show that Φ strongly approximately dominates any map of the form

$$\psi(a) = \sum_{i,j=1}^n T_i^* \phi(a_i^* a a_j) T_j$$

where $T_i \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ and $\phi \in \mathcal{S}$. Pick a sequence (v_k) in $\mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ such that $\lim v_k^* \Phi(a) v_k = \phi(a)$ and $\lim v_k^* T v_k = 0$ for $T \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$. We get that

$$\psi(a) = \lim_{k \rightarrow \infty} \sum_{i,j=1}^n T_i^* v_k^* \Phi(a_i^* a a_j) v_k T_j = \lim_{k \rightarrow \infty} w_k^* \Phi(a) w_k$$

where $w_k = \sum_{i=1}^n \Phi(a_i) v_k T_i$. Since $\|w_k^* T w_k\| \leq \sum_{i,j=1}^n \|T_i\| \|T_j\| \|v_k^* (\Phi(a_i^*) T \Phi(a_j)) v_k\| \rightarrow 0$ for any T , it follows that Φ strongly approximately dominates ψ . This is obvious since condition (i) of Definition 3.2 is trivially satisfied since $\psi(a), w_k \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$.

We will now prove that if $\psi_1, \dots, \psi_n: \mathfrak{A} \rightarrow \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ such that Φ strongly approximately dominates each ψ_i , then Φ strongly approximately dominates $\psi = \sum_{i=1}^n \psi_i$. First fix $h \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ strictly positive, $0 < \epsilon \leq 1$ and a finite subset $F \subset \mathfrak{A}$ containing 1. It suffices to find $v \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ with $\|v^* v\| \leq \|\psi\| + 1$ such that $\|v^* \Phi(a) v - \psi(a)\|, \|v^* h v\| < \epsilon$ for $a \in F$. Note that any v satisfying the above (with no norm consideration) will satisfy $\|v^* v - \psi(1)\| < \epsilon$ and thus $\|v^* v\| \leq \|\psi(1)\| + \epsilon \leq \|\psi\| + 1$. Pick $v_1 \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ such that $\|v_1^* \Phi(a) v_1 - \psi_1(a)\|, \|v_1^* h v_1\| < \epsilon/n^2$ for $a \in F$. Recursively, pick $v_k \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ such that

$$\|v_k^* \Phi(a) v_k - \psi_k(a)\|, \|v_j^* T v_k\|, \|v_k^* h v_k\| < \epsilon/n^2,$$

for $a \in F$, $j = 1, \dots, k-1$, and $T \in \{h\} \cup \Phi(F \cup F^*)$. Letting $v = \sum_{i=1}^n v_i$ we get that

$$\|v^* \Phi(a) v - \psi(a)\| \leq \sum_{i=1}^n \|v_i^* \Phi(a) v_i - \psi_i(a)\| + \sum_{1 \leq j < k \leq n} (\|v_j^* \Phi(a) v_k\| + \|v_k^* \Phi(a) v_j\|) < \epsilon$$

for $a \in F$. Moreover, $\|v^* h v\| \leq \sum_{i,j=1}^n \|v_i^* h v_j\| < \epsilon$ and thus Φ strongly approximately dominates ψ .

Finally, let $\psi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ be weakly in \mathcal{C} . We may assume that ψ is a contraction. Let (Ψ, V) be a unital Kasparov–Stinespring dilation of ψ weakly in \mathcal{C} by Theorem 2.13. It suffices to show that Φ strongly approximately dominates Ψ , since if W_n implements a strong approximate domination of Ψ , then $W_n V$ implements a strong approximate domination of ψ .

By Theorem 3.4 Φ strongly approximate dominates Ψ if and only if Φ strongly approximately dominates $T^* \Psi(-) T$ for any $T \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$. Since $T^* \Psi(-) T$ is in $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$, which is generated by \mathcal{S} , it follows from Remark 2.6 that $T^* \Psi(-) T$ may be approximated point-norm by a sum of maps of the form considered in the first part of the proof. Thus by what we have already proven, Φ strongly approximately dominates $T^* \Psi(-) T$ and thus also ψ .

For the “in particular”, the “if” part follows almost immediately from what has already been proven. In fact, if Ψ is a unital representation weakly in \mathcal{C} , then Φ strongly approximately dominates Ψ and by Theorem 3.4

$$\Phi \sim_{as} \Phi \oplus \Psi_\infty \sim_{as} \Phi \oplus \Psi_\infty \oplus \Psi \sim_{as} \Phi \oplus \Psi.$$

The “only if” part follows from Lemma 3.11. \square

Remark 3.13. As mentioned earlier, the above theorem is in some sense the main ingredient, or trick, in the absorption theorems involving nuclear maps. In fact, if $\mathcal{C} = CP_{\text{nuc}}(\mathfrak{A}, \mathfrak{B})$ is the closed operator convex cone consisting of all nuclear maps, and \mathfrak{B} is stable, then \mathcal{C} is generated by the set

$$\mathcal{S} = \{\mathfrak{A} \xrightarrow{b\rho(-)} \mathfrak{B} : \rho \text{ is a pure state on } \mathfrak{A}, b \in \mathfrak{B}_+\}.$$

This was basically proven by Kirchberg in the pre-print [Kir94], although not stated in this way, and a similar proof can be found in the proof of [EK01, Lemma 10]. Now, if \mathfrak{A} is the quotient of a C^* -subalgebra $\mathfrak{E} \subset \mathcal{M}(\mathfrak{B})$, then any pure state ρ on \mathfrak{A} lifts to a pure state $\tilde{\rho}$ of \mathfrak{E} . If the extension $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ has nice comparison properties (cf. Lemma 6.8), then we may use excision of pure states in \mathfrak{E} to show the inclusion $\iota: \mathfrak{E} \hookrightarrow \mathcal{M}(\mathfrak{B})$ strongly approximately dominates every map in $\mathcal{S}_0 = \{\phi \circ p : \phi \in \mathcal{S}\}$. When this is the case, then ι is \mathcal{C}_0 absorbing by Lemma 2.24 and the above theorem, where $\mathcal{C}_0 = \{\phi \circ p : \phi \in CP_{\text{nuc}}(\mathfrak{A}, \mathfrak{B})\}$. This is essentially what will be done when proving Theorem 6.11.

We can now prove the main theorem on existence of absorbing representations.

Theorem 3.14 (Existence). *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras, with \mathfrak{A} separable and \mathfrak{B} σ -unital, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Then there exists a \mathcal{C} -absorbing representation $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ which is weakly in \mathcal{C} if and only if \mathcal{C} is countably generated.*

Similarly, if \mathfrak{A} is unital and \mathcal{C} is non-degenerate, then there exists a unital \mathcal{C} -absorbing representation $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ which is weakly in \mathcal{C} if and only if \mathcal{C} is countably generated.

Proof. We will prove the unital case first. Since \mathfrak{B} is σ -unital, so is $\mathbb{K}(\mathcal{H}_{\mathfrak{B}})$. Let $T \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$ be strictly positive and suppose that $\Phi: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}})$ is unital \mathcal{C} -absorbing and weakly in \mathcal{C} . We will show that $T\Phi(-)T$ generates $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$ as a closed operator convex cone, i.e. that $K(\{T\Phi(-)T\}) = \mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$.

Let $\psi \in \mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$. By Lemma 3.11 Φ strongly asymptotically dominates ψ and thus ψ can be approximated point-norm by maps of the form $V^*\Phi(-)V$ with $V \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$. Since T is strictly positive any such V may be approximated by TW for some $W \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}})$. Hence ψ can be approximated point-norm by maps of the form $W^*T\Phi(-)TW \in K(\{T^*\Phi(-)T\})$, and since $K(\{T\Phi(-)T\})$ is point-norm closed, it contains ψ . Hence $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}} \subset K(\{T\Phi(-)T\})$ and since the other inclusion is trivial, these two cones are equal. It follows that $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$ is countably generated (in fact, it is singly generated). Thus it follows from Corollary 2.25 that \mathcal{C} is countably generated.

Now suppose \mathcal{C} is countably generated and non-degenerate. Then $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$ is countably generated by Corollary 2.25. Let $(\phi_n)_{n=1}^\infty$ be a sequence of contractive maps in $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$ such that each map is repeated infinitely often in the sequence, and such that the sequence generates $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$. For each $n \in \mathbb{N}$ let (Φ_n, V_n) be a unital Kasparov–Stinespring dilation weakly in \mathcal{C} as in Theorem 2.13, and let $\Phi = \coprod \Phi_n: \mathfrak{A} \rightarrow \mathbb{B}(\mathcal{H}_{\mathfrak{B}}^\infty)$. It follows from Theorem 3.12 that Φ

is unitaly \mathcal{C} absorbing if Φ strongly approximately dominates each ϕ_n . For a fixed $n \in \mathbb{N}$ let $n_1 < n_2 < \dots$ such that $\phi_{n_k} = \phi_n$ for all k . Let $W_k \in \mathbb{B}(\mathcal{H}_{\mathfrak{B}}, \mathcal{H}_{\mathfrak{B}}^\infty)$ be the isometry which is the embedding into the n_k 'th coordinate of $\mathcal{H}_{\mathfrak{B}}^\infty$. Then $(W_{n_k} V_{n_k})^* \Phi(a) W_{n_k} V_{n_k} = \phi_{n_k}(a) = \phi_n(a)$ for all $a \in \mathfrak{A}$ and $\|TW_{n_k} V_{n_k}\| \rightarrow 0$ for all $T \in \mathbb{K}(\mathcal{H}_{\mathfrak{B}}^\infty)$. Hence Φ strongly approximately dominates each ϕ_n which finishes the proof in the unital case.

If Φ is \mathcal{C} -absorbing, then the exact same proof as in the unital case shows that $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$ is singly generated and thus \mathcal{C} is countably generated.

Suppose that \mathcal{C} is countably generated, say by a sequence (ϕ_n) of contractive maps in \mathcal{C} . We will show that \mathcal{C}^\dagger is countably generated, and since this is non-degenerate by Lemma 3.9, it follows from the unital case that there is a unitaly \mathcal{C}^\dagger -absorbing representation Ψ weakly in \mathcal{C}^\dagger . Since $\Psi = \Phi^\dagger$ where $\Phi = \Psi|_{\mathfrak{A}}$, it follows from Proposition 3.10 that Φ is \mathcal{C} -absorbing and weakly in \mathcal{C} thus finishing the proof. So it remains to show that \mathcal{C}^\dagger is countably generated.

By the same argument as above, the sequence $(h\phi_n(-)h)$ also generates \mathcal{C} , where $h \in \mathfrak{B}$ is some strictly positive element. For each n , let $\phi_n^\dagger: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ be the unital extension of ϕ_n which is weakly in \mathcal{C}^\dagger . Clearly $\mathcal{K} := K(\{h\phi_0^\dagger(-)h, h\phi_1^\dagger(-)h, \dots\}) \subset \mathcal{C}^\dagger$ where $\phi_0 = 0$. Let $\psi \in \mathcal{C}^\dagger$ so that we wish to show that $\psi \in \mathcal{K}$. We will apply Theorem 2.18. By split exactness of the short exact sequence

$$0 \rightarrow C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A} \rightarrow C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}^\dagger \rightarrow C^*(\mathbb{F}_\infty) \rightarrow 0,$$

any positive element $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}^\dagger$ may be decomposed into a linear combination of positive elements from $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}$ and $C^*(\mathbb{F}_\infty)$. Hence it suffices to check the condition of Theorem 2.18 for positive elements in $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}$ and in $C^*(\mathbb{F}_\infty)$. If $a \in C^*(\mathbb{F}_\infty) \otimes_{\max} \mathfrak{A}$ then $(id \otimes \psi)(a) = (id \otimes \psi|_{\mathfrak{A}})(a)$ is in the two-sided, closed ideal generated by

$$\{(id \otimes \phi)((1 \otimes x)^* a (1 \otimes x)) : \phi = h\phi_n^\dagger(-)h \text{ for some } n, x \in \mathfrak{A}^\dagger\}$$

since $(h\phi_n(-)h)$ generates \mathcal{C} . If $c \in C^*(\mathbb{F}_\infty) \otimes \mathbb{C}$ is positive, then $(id \otimes \psi)(c) = c \otimes \psi(1)$ is in the two-sided, closed ideal generated by $(id \otimes (h\phi_0^\dagger h))(c) = c \otimes h^2$, (recall that $\phi_0 = 0$). Thus by Theorem 2.18, $\psi \in \mathcal{K}$, which finishes the proof since $\mathcal{C}^\dagger = \mathcal{K}$ is countably generated. \square

We will prove that a closed operator convex cone $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ is countably generated (in fact separable) if \mathfrak{A} and \mathfrak{B} are both separable. This shows, that in most cases of interest, there always exist \mathcal{C} -absorbing representations which are weakly in \mathcal{C} , and unital ones if \mathcal{C} is non-degenerate. Also, this generalises the main result in [Tho01], which corresponds to the case where $\mathcal{C} = CP(\mathfrak{A}, \mathfrak{B})$.

The following result should be well-known.

Lemma 3.15. *Let \mathfrak{A} and \mathfrak{B} be separable C^* -algebras. Then $CP(\mathfrak{A}, \mathfrak{B})$ is second countable in the point-norm topology. In particular, any subspace $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ is separable.*

Proof. As done in the proof of [Tho01, Lemma 2.3], $CP(\mathfrak{A}, \mathfrak{B})$ is metrisable and separable. Thus it is second countable. \square

Since a point-norm separable closed operator convex cone is clearly generated by a countable point-norm dense subset, we may apply the above lemma together with Theorem 3.14 to get the following.

Corollary 3.16. *Let \mathfrak{A} and \mathfrak{B} be separable C^* -algebras, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Then there exists a \mathcal{C} -absorbing representation weakly in \mathcal{C} .*

Moreover, if \mathfrak{A} is unital and \mathcal{C} is non-degenerate, then there exists a unital \mathcal{C} -absorbing representation weakly in \mathcal{C} .

Remark 3.17. Every result in this section has an analogue in the multiplier algebra picture, but we need some notation to see this. If $\mathcal{M}(\mathfrak{B})$ contains \mathcal{O}_2 -isometries s_1, s_2 , i.e. s_1 and s_2 are isometries such that $s_1 s_1^* + s_2 s_2^* = 1_{\mathcal{M}(\mathfrak{B})}$, then we define the Cuntz sum \oplus_{s_1, s_2} in $\mathcal{M}(\mathfrak{B})$ to be $a \oplus_{s_1, s_2} b = s_1 a s_1^* + s_2 b s_2^*$. In particular, this construction can be made when \mathfrak{B} is stable. Such a Cuntz sum is unique up to unitary equivalence. In fact, if t_1, t_2 are also \mathcal{O}_2 -isometries, then $u = t_1 s_1^* + t_2 s_2^*$ is a unitary such that $u^*(a \oplus_{t_1, t_2} b)u = a \oplus_{s_1, s_2} b$. Hence we will often refer to a Cuntz sum as *the* Cuntz sum and simply write \oplus instead of \oplus_{s_1, s_2} . It is not hard to see that, up to some canonical isomorphisms, Cuntz sums correspond to identifying $\mathcal{H}_{\mathfrak{B}} \oplus \mathcal{H}_{\mathfrak{B}} \cong \mathcal{H}_{\mathfrak{B}}$ by some unitary.

Similarly, when \mathfrak{B} is stable, there is an analogue of identifying $\mathcal{H}_{\mathfrak{B}}^\infty \cong \mathcal{H}_{\mathfrak{B}}$ by some unitary. To obtain this analogue, pick a sequence of isometries t_1, t_2, \dots in $\mathcal{M}(\mathfrak{B})$ such that $\sum_{k=1}^\infty t_k t_k^* = 1$, convergence in the strict topology. Note that existence of such a sequence is equivalent to stability of \mathfrak{B} . A bounded sequence of elements (b_n) in $\mathcal{M}(\mathfrak{B})$, which corresponds to a diagonal element in $\mathbb{B}(\mathcal{H}_{\mathfrak{B}}^\infty)$, then corresponds to $\sum_{n=1}^\infty t_n b_n t_n^*$ in $\mathcal{M}(\mathfrak{B})$. An argument as above shows that this identification is unique up to unitary equivalence. In particular, the infinite repeat of an element $m \in \mathcal{M}(\mathfrak{B})$ will be $\sum_{n=1}^\infty t_n m t_n^*$.

Thus, in the multiplier algebra picture, we obtain complete analogues of all results in this section.

We will end this section by applying the above remark to show, that if \mathfrak{A} is separable, and \mathfrak{B} is σ -unital and stable, then any countably generated, closed operator convex cone is singly generated in an exceptionally nice way.

Corollary 3.18. *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras, with \mathfrak{A} separable and \mathfrak{B} is σ -unital and stable, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a countably generated, closed operator convex cone. Then there exists a c.p. map $\phi \in \mathcal{C}$ such that \mathcal{C} is the point-norm closure of the set*

$$\{b^* \phi(-) b : b \in \mathfrak{B}\}.$$

Proof. By Theorem 3.14 and Remark 3.17 there is a \mathcal{C} -absorbing $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ which is weakly in \mathcal{C} . Let $h \in \mathfrak{B}$ be strictly positive and $\phi = h\Phi(-)h$. It follows from Lemma 3.11 that for any $\psi \in \mathcal{C}$ there is $b_0 \in \mathfrak{B}$ such that $b_0^* \Phi(-) b_0$ is approximately ψ . Since b_0 may be approximated by hb for some $b \in \mathfrak{B}$, it follows that $b^* \phi(-) b$ approximates ψ . \square

4. THE PURELY LARGE PROBLEM

When trying to determine when representations are absorbing, it is often useful to consider the induced extension of C^* -algebras. This was the strategy of Elliott and Kucerovsky in [EK01], in which they prove that a unital $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ absorbs any unital weakly nuclear $*$ -homomorphism exactly when the extension induced by Φ is *purely large*.

Motivated by this, we turn our study to extensions of C^* -algebras with respect to closed operator convex cones. In doing this, it seems more natural to work in the multiplier

algebra picture of the theory, which we will thus do. As mentioned in Remark 3.17, all results in the previous section can be defined in the multiplier algebra picture as well. Here $\mathcal{C}_{\mathcal{H}_{\mathfrak{B}}}$ corresponds to \mathcal{C}^s , as in Proposition 2.17, and if \mathfrak{B} is stable it corresponds to \mathcal{C} by Corollary 2.23.

Recall, that a short exact sequence of C^* -algebras $\mathfrak{e} : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \xrightarrow{p} 0$ is called an *extension* of \mathfrak{A} by \mathfrak{B} . Given such an extension there is an induced $*$ -homomorphism $\tau : \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$ called the *Busby map*, where $\mathcal{Q}(\mathfrak{B}) = \mathcal{M}(\mathfrak{B})/\mathfrak{B}$ is the corona algebra. By the universal property of multiplier algebras, there is a canonical $*$ -homomorphism $\sigma : \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$. The induced $*$ -homomorphism to the pull-back $(p, \sigma) : \mathfrak{E} \rightarrow \mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B})$ is an isomorphism. Thus when studying extensions of \mathfrak{A} by \mathfrak{B} , one might as well study $*$ -homomorphisms $\tau : \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$.

Given two Busby maps $\tau_1, \tau_2 : \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$ we say that τ_1 and τ_2 are *equivalent*, written $\tau_1 \sim \tau_2$, if there is a unitary $u \in \mathcal{M}(\mathfrak{B})$ such that $\tau_1 = \text{Ad } \pi(u) \circ \tau_2$. Such a unitary implementing an equivalence of Busby maps induces an isomorphism of the corresponding extensions.

When $\mathcal{M}(\mathfrak{B})$ contains a unital copy of \mathcal{O}_2 (e.g. when \mathfrak{B} is stable), then we may define Cuntz sums of Busby maps by $\tau_1 \oplus_{s_1, s_2} \tau_2 = \pi(s_1)\tau_1(-)\pi(s_1)^* + \pi(s_2)\tau_2(-)\pi(s_2)^*$ where s_1, s_2 are \mathcal{O}_2 -isometries in $\mathcal{M}(\mathfrak{B})$, i.e. isometries satisfying $s_1 s_1^* + s_2 s_2^* = 1$. As in Remark 3.17 the Cuntz sum is unique up to equivalence of Busby maps, thus we will often refer to *the* Cuntz sum.

We say τ_1 *absorbs* τ_2 (or that the corresponding extension \mathfrak{e}_1 *absorbs* \mathfrak{e}_2) if $\tau_1 \oplus \tau_2 \sim \tau_1$.

Definition 4.1. Let $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras and $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. If \mathfrak{E} is unital (resp. not necessarily unital) we say that the extension is a *unital \mathcal{C} -extension* (resp. *\mathcal{C} -extension*) if it has a unital c.p. splitting (resp. a c.p. splitting) which is weakly in \mathcal{C} .

Moreover, if the splitting can be chosen to be a $*$ -homomorphism, then we say that the extension is a *trivial, unital \mathcal{C} -extension* (resp. *trivial \mathcal{C} -extension*).

Note that by assumption any \mathcal{C} -extension (even when $\mathcal{C} = CP(\mathfrak{A}, \mathfrak{B})$) is assumed to have a c.p. splitting.

Remark 4.2. A trivial, unital \mathcal{C} -extension is always assumed to have a unital splitting $*$ -homomorphism. Thus if we have a unital \mathcal{C} -extension which has a splitting $*$ -homomorphism which is not necessarily unital, then it is a trivial \mathcal{C} -extension, which just happens to be unital, but it is *not necessarily* a trivial, unital \mathcal{C} -extension. So there is a difference between a trivial, unital \mathcal{C} -extension and a unital, trivial \mathcal{C} -extension. However, in this paper we never consider unital, trivial \mathcal{C} -extensions (unless our trivial \mathcal{C} -extension simply happens to be unital, in which case we do not mention unitality). So whenever the reader sees the words “trivial” and “unital” near “ \mathcal{C} -extension”, we will always assume that the splitting $*$ -homomorphism weakly in \mathcal{C} can be chosen to be unital.

As defined in [EK01], we say that an extension $\mathfrak{e} : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ of C^* -algebras is *purely large* if for every $x \in \mathfrak{E} \setminus \mathfrak{B}$, there is a σ -unital, stable C^* -subalgebra $\mathfrak{D} \subset \overline{x^* \mathfrak{B} x}$ which is full in \mathfrak{B} . Note that we have added the condition that \mathfrak{D} be σ -unital. This was redundant in [EK01] since they almost always assume that \mathfrak{B} is separable.

Recall the main result of [EK01]. Note that we have changed this in accordance to the correction made by the first named author in [Gab14].

Theorem 4.3 ([EK01], Theorem 6 and Corollary 16). *Let $\mathfrak{e} : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{C} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of separable C^* -algebras, with \mathfrak{B} stable, and let $\mathcal{C} = CP_{\text{nuc}}(\mathfrak{A}, \mathfrak{B})$. Then \mathfrak{e} absorbs any trivial \mathcal{C} -extension if and only if \mathfrak{e} absorbs the zero extension and is purely large.*

Moreover, if \mathfrak{e} is unital, then \mathfrak{e} absorbs any trivial, unital \mathcal{C} -extension if and only if \mathfrak{e} is purely large.

As we will see as a special case of Theorem 6.11 below, the above theorem also holds when \mathfrak{B} is only assumed to be σ -unital.

In this section we address the question, of whether or not we can get a condition similar to extensions being purely large, for extensions with a c.p. splitting which is weakly in \mathcal{C} . We will use the following notation.

Notation 4.4. Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. We let $\mathbb{I}(\mathfrak{B})$ denote the complete lattice of two-sided, closed ideals in \mathfrak{B} , and let $\mathfrak{B}_{\mathcal{C}} : \mathfrak{A} \rightarrow \mathbb{I}(\mathfrak{B})$ denote the map given by

$$\mathfrak{B}_{\mathcal{C}}(a) = \overline{\mathfrak{B}\{\phi(a) : \phi \in \mathcal{C}\}\mathfrak{B}}.$$

Example 4.5. Let \mathcal{C} denote either $CP(\mathfrak{A}, \mathfrak{B})$ or $CP_{\text{nuc}}(\mathfrak{A}, \mathfrak{B})$. Then

$$\mathfrak{B}_{\mathcal{C}}(a) = \begin{cases} \mathfrak{B}, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0. \end{cases}$$

Example 4.6. As mentioned after Definition 2.12, if \mathfrak{A} has a strictly positive element h , then \mathcal{C} is non-degenerate if and only if $\mathfrak{B}_{\mathcal{C}}(h) = \mathfrak{B}$.

A somewhat surprising result of Kirchberg and Rørdam [KR05, Proposition 4.2] says that a closed operator convex cone $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$, where \mathfrak{A} is separable and nuclear, is uniquely determined by the map $\mathfrak{B}_{\mathcal{C}}$. This suggests that any closed operator convex cone consists of some given relation between the ideal structures of \mathfrak{A} and \mathfrak{B} and some nuclearity conditions on the maps. However, we will not use their result in this paper, since we almost only consider C^* -algebras which are not necessarily nuclear.

Lemma 4.7. *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras, with \mathfrak{A} separable, \mathfrak{B} σ -unital and stable, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a countably generated, closed operator convex cone. If $\Phi : \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is a \mathcal{C} -absorbing $*$ -homomorphism weakly in \mathcal{C} , then*

$$\mathfrak{B}_{\mathcal{C}}(a) = \overline{\mathfrak{B}\Phi(a)\mathfrak{B}}$$

for every $a \in \mathfrak{A}$.

Moreover, if \mathfrak{A} is unital, \mathcal{C} is non-degenerate and $\Phi : \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is a unital \mathcal{C} -absorbing $$ -homomorphism weakly in \mathcal{C} , then*

$$\mathfrak{B}_{\mathcal{C}}(a) = \overline{\mathfrak{B}\Phi(a)\mathfrak{B}}$$

for every $a \in \mathfrak{A}$.

Proof. This follows from Lemma 3.11. □

Remark 4.8. It can easily be seen by the above lemma, that if a_1 and a_2 in \mathfrak{A} generate the same two-sided, closed ideal, then $\mathfrak{B}_{\mathcal{C}}(a_1) = \mathfrak{B}_{\mathcal{C}}(a_2)$ if \mathfrak{A} , \mathfrak{B} and \mathcal{C} satisfy the (modest) assumptions of the lemma. Hence the map $\mathfrak{B}_{\mathcal{C}} : \mathfrak{A} \rightarrow \mathbb{I}(\mathfrak{B})$ drops to an action $\mathbb{O}(\text{Prim } \mathfrak{A}) \cong \mathbb{I}(\mathfrak{A}) \rightarrow \mathbb{I}(\mathfrak{B})$ as described in Section 5.1. Thus any such closed operator convex cone induces an action of $\text{Prim } \mathfrak{A}$ on \mathfrak{B} which uniquely determines $\mathfrak{B}_{\mathcal{C}}$.

We give a generalised description of purely large extensions as follows.

Definition 4.9. Let $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras, with \mathfrak{A} separable, \mathfrak{B} σ -unital and stable, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a countably generated, closed operator convex cone. We will say that the extension is \mathcal{C} -purely large if for any $x \in \mathfrak{E}$, $\overline{x^*\mathfrak{B}x}$ contains a σ -unital and stable sub- C^* -algebra \mathfrak{D} which is full in $\mathfrak{B}_{\mathcal{C}}(p(x))$.

Since $\overline{x^*\mathfrak{B}x} = \overline{x^*x\mathfrak{B}x^*x}$ and since $\mathfrak{B}_{\mathcal{C}}(p(x)) = \mathfrak{B}_{\mathcal{C}}(p(x^*x))$ by Lemma 4.7, it suffices to check the above condition only for positive x .

Note that an extension being \mathcal{C} -purely large only depends on the map $\mathfrak{B}_{\mathcal{C}}$.

The case above when the map $\mathfrak{B}_{\mathcal{C}}$ is the map defined in Example 4.5, corresponds to the purely large condition defined in [EK01].

The following proposition, together with the main result of [EK01], motivates the purely large problem.

Proposition 4.10. Let $\mathfrak{e} : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras, with \mathfrak{A} separable, \mathfrak{B} σ -unital and stable, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a countably generated, closed operator convex cone. If \mathfrak{e} absorbs any trivial \mathcal{C} -extension, then the extension is \mathcal{C} -purely large.

Moreover, the same holds if \mathfrak{e} is unital, \mathcal{C} is non-degenerate, and \mathfrak{e} absorbs any trivial, unital \mathcal{C} -extension.

Proof. The proof is very similar to that of [EK01, Theorem 17(iii)]. We will give the argument for completion and since it is not identical to their proof. We will only prove the non-unital case, since the unital case is identical.

Let τ be the Busby map of \mathfrak{e} and let $\Phi_{\mathcal{C}} : \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ denote a \mathcal{C} -absorbing $*$ -homomorphism weakly in \mathcal{C} of Theorem 3.14. Fix isometries t_1, t_2, \dots such that $\sum_{k=1}^{\infty} t_k t_k^* = 1$. Let $\Phi = \sum_{k=1}^{\infty} t_k \Phi_{\mathcal{C}}(-) t_k^*$ denote the infinite repeat, and \mathfrak{e}' denote the extension with Busby map $\tau' = \pi \circ \Phi$. Since Φ is weakly in \mathcal{C} , \mathfrak{e} absorbs \mathfrak{e}' . Thus, given \mathcal{O}_2 -isometries $s_1, s_2 \in \mathcal{M}(\mathfrak{B})$, we may assume without loss of generality that \mathfrak{e} has Busby map $\tau \oplus_{s_1, s_2} \tau'$.

Let $\sigma : \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ be the canonical map and let $x \in \mathfrak{E}$ so that we should show the condition in Definition 4.9. We may assume that $\|x\| \leq 1$, and as remarked after the definition, we may also assume that x is positive. Let $P = 0 \oplus_{s_1, s_2} 1 \in \mathcal{M}(\mathfrak{B})$ and $\Phi' = 0 \oplus_{s_1, s_2} \Phi$. Define $m = \sigma(x)^{1/2} P \sigma(x)^{1/2} \in \mathcal{M}(\mathfrak{B})$, $a = p(x)$ and note that $m - \Phi'(a) \in \mathfrak{B}$. We get that $0 \leq m' := m^{1/2} \Phi'(a) m^{1/2} \leq m \leq \sigma(x)$ and $0 \leq m'' := \Phi'(a)^{1/2} m \Phi'(a)^{1/2} \leq \Phi'(a)$. Set $\mathfrak{D} := \overline{m' \mathfrak{B} m'} \subseteq \overline{x \mathfrak{B} x}$ and note that it is σ -unital as \mathfrak{B} is σ -unital, and that $\mathfrak{D} \cong \overline{m'' \mathfrak{B} m''}$. Hence it suffices to show that \mathfrak{D} is full in $\mathfrak{B}_{\mathcal{C}}(a)$ and that $\overline{m'' \mathfrak{B} m''}$ is stable.

By Lemma 4.7

$$\mathfrak{B}_{\mathcal{C}}(a) = \overline{\mathfrak{B} \Phi_{\mathcal{C}}(a) \mathfrak{B}} = \overline{\mathfrak{B} \Phi'(a) \mathfrak{B}} \supseteq \overline{\mathfrak{B} \mathfrak{D} \mathfrak{B}}.$$

Note that $m' - \Phi'(a)^2 \in \mathfrak{B}$ and thus $(m' - \Phi'(a)^2)(0 \oplus_{s_1, s_2} t_k) \rightarrow 0$. Any positive element in $\mathfrak{B}_{\mathcal{C}}(a)$ is close to an element of the form

$$\begin{aligned} \sum_{i=1}^n b_i^* (0 \oplus_{s_1, s_2} \Phi_{\mathcal{C}}(a)^2) b_i &= \sum_{i=1}^n b_i^* (0 \oplus_{s_1, s_2} t_k^* t_k \Phi_{\mathcal{C}}(a)^2 t_k^* t_k) b_i \\ &= \sum_{i=1}^n b_i^* (0 \oplus_{s_1, s_2} t_k^*) \Phi'(a)^2 (0 \oplus_{s_1, s_2} t_k) b_i \\ &\approx \sum_{i=1}^n b_i^* (0 \oplus_{s_1, s_2} t_k^*) m' (0 \oplus_{s_1, s_2} t_k) b_i \in \overline{\mathfrak{B} \mathfrak{D} \mathfrak{B}}, \end{aligned}$$

for large k , where $b_1, \dots, b_n \in \mathfrak{B}$. Hence \mathfrak{D} is full in $\mathfrak{B}_{\mathcal{C}}(a)$.

It remains to show that $\mathfrak{D} \cong \overline{m'' \mathfrak{B} m''}$ is stable. By [HR98, Theorem 2.1 and Proposition 2.2] it suffices to show that for any positive $b \in \overline{m'' \mathfrak{B} m''}$, there is a sequence (b_k) of elements in $\overline{m'' \mathfrak{B} m''}$ such that $b_k^* b_k - b \rightarrow 0$ and $(b_k^* b_k)(b_k b_k^*) \rightarrow 0$.

Fix a positive $b \in \overline{m'' \mathfrak{B} m''} \subseteq \overline{\Phi'(a) \mathfrak{B} \Phi'(a)}$. Note that $\Phi'(a)^{2/k} b \rightarrow b$ and that $(m'')^{1/k} - \Phi'(a)^{2/k} \in \mathfrak{B}$ for all k . As in [EK01, Theorem 17 (iii)], which is basically a trick from [HR98], we may pick a sequence of unitaries (u_n) in $\mathcal{M}(\mathfrak{B})$ such that each u_n commutes with Φ' , and $b_1 u_n b_2 \rightarrow 0$ for $b_1, b_2 \in \mathfrak{B}$. For each k let n_k be such that

$$(m'')^{1/k} u_{n_k} b^{1/2} \approx_{1/k} \Phi'(a)^{2/k} u_{n_k} b^{1/2} = u_{n_k} \Phi'(a)^{2/k} b^{1/2}.$$

Let $b_k := (m'')^{1/k} u_{n_k} b^{1/2} \in \overline{m'' \mathfrak{B} m''}$ and note that the sequence (b_k) is bounded. We have that $b_k - u_{n_k} b^{1/2} \rightarrow 0$ and thus $b_k^* b_k - b \rightarrow 0$ since (b_k) is bounded. Since

$$\begin{aligned} (b_k^* b_k)(b_k b_k^*) &= b_k^* (m'')^{1/k} u_{n_k} b^{1/2} (m'')^{1/k} u_{n_k} b^{1/2} b_k^* \\ &\approx b_k^* (m'')^{1/k} u_{n_k} (b^{1/2} u_{n_k} b^{1/2}) b_k^* \quad (\text{for large } k) \\ &\rightarrow 0 \end{aligned}$$

it follows that $\overline{m'' \mathfrak{B} m''}$ is stable by [HR98, Theorem 2.1 and Proposition 2.2]. \square

We raise the following question.

Question 4.11 (The purely large problem). Let \mathfrak{A} and \mathfrak{B} be C^* -algebras with \mathfrak{A} separable and \mathfrak{B} stable and σ -unital. For which countably generated, closed operator convex cones $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ do the following hold? For any extension \mathfrak{e} of \mathfrak{A} by \mathfrak{B} :

- (Q1) \mathfrak{e} absorbs any trivial \mathcal{C} -extension if and only if \mathfrak{e} absorbs the zero extension and is \mathcal{C} -purely large,
- (Q2) If, in addition, \mathfrak{e} is unital and \mathcal{C} is non-degenerate, then \mathfrak{e} absorb any trivial, unital \mathcal{C} -extension if and only if \mathfrak{e} is \mathcal{C} -purely large.

Definition 4.12. Let \mathfrak{A} , \mathfrak{B} and \mathcal{C} be as in Question 4.11 above. We say that \mathcal{C} *satisfies the purely large problem* if it satisfies (Q1) above.

If, in addition, \mathfrak{A} is unital and \mathcal{C} is non-degenerate, then we say that \mathcal{C} *satisfies the unital purely large problem* if it satisfies (Q2) above.

Using the above definition, one could reformulate Theorem 4.3 as saying that if \mathfrak{A} and \mathfrak{B} are separable C^* -algebras with \mathfrak{B} stable (and \mathfrak{A} unital), then $CP_{\text{nuc}}(\mathfrak{A}, \mathfrak{B})$ satisfies the (unital) purely large problem.

An affirmative answer to the purely large problem implies a Weyl–von Neumann type theorem as follows.

Proposition 4.13 (Abstract Weyl–von Neumann theorem). *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras with \mathfrak{A} separable and \mathfrak{B} stable and σ -unital. Suppose that $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ is a countably generated, closed operator convex cone. If \mathcal{C} satisfies the purely large problem, then a $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is \mathcal{C} -absorbing if and only if the extension with Busby map $\pi \circ \Phi$ is \mathcal{C} -purely large and absorbs the zero extension.*

If \mathfrak{A} is unital, \mathcal{C} is non-degenerate and satisfies the unital purely large problem, then a unital $$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is unittally \mathcal{C} -absorbing if and only if the extension with Busby map $\pi \circ \Phi$ is \mathcal{C} -purely large.*

Proof. If Φ is (unittally) \mathcal{C} -absorbing then clearly the extension \mathfrak{e} with Busby map $\pi \circ \Phi$ absorbs every trivial, (unital) \mathcal{C} -extension. By Proposition 4.10, \mathfrak{e} is \mathcal{C} -purely large. In the non-unital case, Φ absorbs the zero homomorphism, so \mathfrak{e} absorbs the zero extension.

We first prove the unital case. Suppose that $\pi \circ \Phi$ is \mathcal{C} -purely large and thus absorbs any trivial, unital \mathcal{C} -extension. Let $\Psi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ be a unital $*$ -homomorphism weakly in \mathcal{C} , and let Ψ_∞ be an infinite repeat. Then there is a unitary $U \in \mathcal{M}(\mathfrak{B})$ such that $U^*(\Phi(a) \oplus \Psi_\infty(a))U - \Phi(a) \in \mathfrak{B}$ for all $a \in \mathfrak{A}$. By Theorem 3.4, Φ absorbs Ψ_∞ and thus it also absorbs Ψ , so Φ is unittally \mathcal{C} -absorbing.

In the non-unital case, let $\Psi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ be a $*$ -homomorphism weakly in \mathcal{C} . As above we may find a unitary $U \in \mathcal{M}(\mathfrak{B})$ such that $U^*(\Phi(a) \oplus \Psi_\infty(a))U - \Phi(a) \in \mathfrak{B}$ for all $a \in \mathfrak{A}$. As in the proof of Proposition 3.10 it follows that $U^*(\Phi^\dagger(a) \oplus \Psi_\infty^\dagger(a))U - \Phi^\dagger(a) \in \mathfrak{B}$ for all $a \in \mathfrak{A}^\dagger$. As above, Φ^\dagger absorbs Ψ^\dagger which implies that Φ absorbs Ψ , so Φ is \mathcal{C} -absorbing. \square

Remark 4.14. The purely large problem does not always have an affirmative answer. If \mathcal{C} satisfies the (unital) purely large problem and \mathcal{C}' is some other closed operator convex cone with $\mathfrak{B}_{\mathcal{C}} = \mathfrak{B}_{\mathcal{C}'}$, then \mathcal{C}' does *not* satisfy the (unital) purely large problem.

To see this, suppose that both \mathcal{C} and \mathcal{C}' satisfy the (unital) purely large problem, and that $\mathfrak{B}_{\mathcal{C}} = \mathfrak{B}_{\mathcal{C}'}$. Let $\Phi_{\mathcal{C}}, \Phi_{\mathcal{C}'}: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ be the (unital) \mathcal{C} -absorbing and \mathcal{C}' -absorbing $*$ -homomorphisms weakly in \mathcal{C} and \mathcal{C}' respectively. The extensions with Busby maps $\pi \circ \Phi_{\mathcal{C}}$ and $\pi \circ \Phi_{\mathcal{C}'}$ are both \mathcal{C} -purely large and \mathcal{C}' -purely large, since $\mathfrak{B}_{\mathcal{C}} = \mathfrak{B}_{\mathcal{C}'}$. Thus it follows that $\Phi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}'}$ absorb each other, and thus $\mathcal{C} = \mathcal{C}'$.

In particular, if \mathfrak{A} is a separable, stable, non-nuclear C^* -algebra, then $CP(\mathfrak{A}, \mathfrak{A})$ does not satisfy the purely large problem by Example 4.5 and Theorem 4.3.

We will prove that if X is a finite space, \mathfrak{A} and \mathfrak{B} are sufficiently nice X - C^* -algebras (e.g. if both are separable C^* -algebras over X) and $\mathcal{C} = CP_{\text{r-nuc}}(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ then \mathcal{C} satisfies the purely large problem. If, in addition, \mathfrak{A} is unital and \mathcal{C} is non-degenerate, then \mathcal{C} satisfies the unital purely large problem. See Theorem 6.11.

As an additional, or perhaps even preliminary, question to the purely large problem, one can ask the following.

Question 4.15. Is there an easy way to compute the map $\mathfrak{B}_{\mathcal{C}}$?

In the case described above, we have an affirmative answer to this question, see Proposition 6.3. A much more general solution to this question, in many cases of interest, will appear in [Gab15] by the first author.

It turns out that the purely large problem can always be answered by answering a related *unital* purely large problem. We will need the following lemma.

Lemma 4.16. *Let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. If $a \in \mathfrak{A} \subset \mathfrak{A}^\dagger$ then $\mathfrak{B}_{\mathcal{C}^\dagger}(a) = \mathfrak{B}_{\mathcal{C}}(a)$. If $a \in \mathfrak{A}^\dagger \setminus \mathfrak{A}$ then $\mathfrak{B}_{\mathcal{C}^\dagger}(a) = \mathfrak{B}$.*

Proof. Suppose $a \in \mathfrak{A}$. Clearly $\mathfrak{B}_{\mathcal{C}^\dagger}(a) \subset \mathfrak{B}_{\mathcal{C}}(a)$. Let $\phi \in \mathcal{C}$ and (a_λ) be an approximate identity in \mathfrak{A} . Let $\phi_\lambda \in \mathcal{C}^\dagger$ be given by $\phi_\lambda(x + \mu 1) = \phi(a_\lambda x a_\lambda + \mu a_\lambda^2)$. Then $\phi_\lambda|_{\mathfrak{A}} \rightarrow \phi$ point-norm. It follows that $\phi(a) \in \mathfrak{B}_{\mathcal{C}^\dagger}(a)$ for all $\phi \in \mathcal{C}$ and thus $\mathfrak{B}_{\mathcal{C}}(a) \subset \mathfrak{B}_{\mathcal{C}^\dagger}(a)$.

Let $a \in \mathfrak{A}^\dagger \setminus \mathfrak{A}$. For every positive $b \in \mathfrak{B}$ there is a map ϕ_b in \mathcal{C}^\dagger given as the composition

$$\mathfrak{A}^\dagger \twoheadrightarrow \mathfrak{A}^\dagger/\mathfrak{A} \cong \mathbb{C} \xrightarrow{\rho_b} \mathfrak{B}$$

where $\rho_b(\mu) = \mu b$. Since $\phi_b(a) = \mu b$ for some non-zero $\mu \in \mathbb{C}$, it follows that $b \in \mathfrak{B}_{\mathcal{C}^\dagger}(a)$ for every positive $b \in \mathfrak{B}$. Thus $\mathfrak{B}_{\mathcal{C}^\dagger}(a) = \mathfrak{B}$. \square

Proposition 4.17. *Let $\mathfrak{e} : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras, with \mathfrak{A} separable and \mathfrak{B} stable and σ -unital, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a countably generated, closed operator convex cone. If \mathfrak{e} is \mathcal{C} -purely large and absorbs the zero extension, then the unitised extension $\mathfrak{e}^\dagger : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E}^\dagger \xrightarrow{p^\dagger} \mathfrak{A}^\dagger \rightarrow 0$ is \mathcal{C}^\dagger -purely large.*

In particular, if \mathcal{C}^\dagger satisfies the unital purely large problem then \mathcal{C} satisfies the purely large problem.

Proof. Since \mathfrak{e} absorbs the zero extension we may assume that the Busby map τ of \mathfrak{e} is of the form $\tau = \tau_0 \oplus_{s_1, s_2} 0$ for some \mathcal{O}_2 -isometries $s_1, s_2 \in \mathcal{M}(\mathfrak{B})$. Let $x \in \mathfrak{E}^\dagger$ so that we must show that $\overline{x^* \mathfrak{B} x}$ contains a stable, σ -unital C^* -subalgebra \mathfrak{D} which is full in $\mathfrak{B}_{\mathcal{C}^\dagger}(p^\dagger(x))$. If $x \in \mathfrak{E}$, then $\mathfrak{B}_{\mathcal{C}^\dagger}(p^\dagger(x)) = \mathfrak{B}_{\mathcal{C}}(p(x))$ by Lemma 4.16, and thus such a \mathfrak{D} exists, since \mathfrak{e} is \mathcal{C} -purely large. So it remains to check this when $x \in \mathfrak{E}^\dagger \setminus \mathfrak{E}$. Clearly, it suffices to do this when $x = 1 + y$ with $y \in \mathfrak{E}$. Let $\sigma : \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ be the canonical $*$ -homomorphism and let $P = 0 \oplus_{s_1, s_2} 1 \in \mathcal{M}(\mathfrak{B})$. Since $\pi(P\sigma(y)) = \pi(P)\tau(p(y)) = 0 \in \mathcal{Q}(\mathfrak{B})$ it follows that $b := P\sigma(y) \in \mathfrak{B}$. Note that $\mathfrak{B}_{\mathcal{C}^\dagger}(p^\dagger(1 + y)) = \mathfrak{B}$ by Lemma 4.16. Since

$$\overline{(P + b)^* \mathfrak{B} (P + b)} = \overline{(1 + y)^* P \mathfrak{B} P (1 + y)} \subset \overline{(1 + y)^* \mathfrak{B} (1 + y)}$$

it suffices to show that $\overline{(P + b)^* \mathfrak{B} (P + b)}$ contains a stable, σ -unital C^* -subalgebra which is full in \mathfrak{B} . Consider the trivial extension \mathfrak{e}_0 of \mathbb{C} by \mathfrak{B} with splitting $1 \mapsto P$. This extension clearly absorbs any trivial extension of \mathbb{C} by \mathfrak{B} , since the Cuntz sum of P with any projection is unitarily equivalent to P . By Proposition 4.10 it follows that \mathfrak{e}_0 is $CP(\mathbb{C}, \mathfrak{B})$ -purely large, i.e. purely large in the classical sense. Since $P + b$ is an element of the extension algebra of \mathfrak{e}_0 , it follows that $\overline{(P + b)^* \mathfrak{B} (P + b)}$ contains a stable, σ -unital C^* -subalgebra which is full in \mathfrak{B} . Hence \mathfrak{e}^\dagger is \mathcal{C}^\dagger -purely large.

Now, for the “in particular” part, let \mathfrak{f} be a trivial \mathcal{C} -extension. Then \mathfrak{f}^\dagger is a trivial, unital \mathcal{C}^\dagger -extension and is thus absorbed by \mathfrak{e}^\dagger , since \mathcal{C}^\dagger satisfies the unital purely large problem. It easily follows that \mathfrak{e} absorbs \mathfrak{f} , and thus \mathcal{C} satisfies the purely large problem. \square

We end this section by applying results of Kirchberg and Rørdam to give a lot of non-trivial examples of cones satisfying the (unital) purely large problem.

Proposition 4.18. *Let \mathfrak{A} be a separable, nuclear C^* -algebra and \mathfrak{B} be a separable, stable, nuclear, strongly purely infinite C^* -algebra. Then any closed operator convex cone $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ satisfies the purely large problem.*

If, in addition, \mathfrak{A} is unital, then any non-degenerate, closed operator convex cone $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ satisfies the unital purely large problem.

Proof. By Proposition 4.17 it suffices to prove the unital case. By [Kir03, Theorem 6.3] $\mathcal{M}(\mathfrak{B})$ is strongly purely infinite. For any unital \mathcal{C} -purely large extension $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ let $\sigma: \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ be the induced $*$ -homomorphism. Then $\mathfrak{C} := \sigma(\mathfrak{E})$ is a separable, nuclear C^* -subalgebra of $\mathcal{M}(\mathfrak{B})$, and by [KR02, Theorem 7.21] any approximately inner map $\mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ is approximately one-step inner.

We claim that \mathcal{C} -purely largeness implies that $\mathfrak{B}_{\mathcal{C}}(p(x)) \subset \overline{\mathfrak{B}x\mathfrak{B}}$ for any positive $x \in \mathfrak{E}$. To see this, let $\mathfrak{D} \subset \overline{x\mathfrak{B}x}$ be a σ -unital, stable C^* -subalgebra which is full in $\mathfrak{B}_{\mathcal{C}}(p(x))$. Then

$$\mathfrak{B}_{\mathcal{C}}(p(x)) = \overline{\mathfrak{B}\mathfrak{D}\mathfrak{B}} \subset \overline{\mathfrak{B}x\mathfrak{B}x\mathfrak{B}} = \overline{\mathfrak{B}x\mathfrak{B}} = \overline{\mathfrak{B}\sigma(x)\mathfrak{B}}.$$

It follows from [KR05, Corollary 4.3] that any map of the form $\phi \circ p$ with $\phi \in \mathcal{C}$ is approximately inner with respect to σ , and thus also approximately one-step inner, i.e. for any $\phi \in \mathcal{C}$, the c.p. map $\phi \circ p$ is approximated point-norm by maps of the form $m^*\sigma(-)m$ for $m \in \mathcal{M}(\mathfrak{B})$. It easily follows that σ strongly approximately dominates any map $\phi \circ p: \mathfrak{E} \rightarrow \mathfrak{B}$ where $\phi \in \mathcal{C}$. Thus it follows from Theorem 3.12 that σ absorbs $\Phi \circ p$ where $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is a unital \mathcal{C} -absorbing $*$ -homomorphism weakly in \mathcal{C} . It follows that τ absorbs $\pi \circ \Phi$ and thus τ absorbs any trivial, unital \mathcal{C} -extension. \square

Remark 4.19. Note that we may assume something much weaker than \mathcal{C} -purely largeness in the above proposition. In fact, the only thing that \mathcal{C} -purely largeness is used for, is that $\mathfrak{B}_{\mathcal{C}}(p(x)) \subset \overline{\mathfrak{B}x\mathfrak{B}}$ for every positive $x \in \mathfrak{E}$. Hence this may be assumed above instead of \mathcal{C} -purely largeness of the extensions.

In the (classical) case where \mathfrak{B} in addition is simple and $\mathcal{C} = CP(\mathfrak{A}, \mathfrak{B})$, this is the same as noting that an extension is purely large exactly when it is essential. When \mathfrak{B} is simple and σ -unital, then every essential extension by \mathfrak{B} is purely large if and only if \mathfrak{B} is either purely infinite or isomorphic to \mathbb{K} . This result will be generalised in Theorem 7.5.

5. ACTIONS OF TOPOLOGICAL SPACES ON C^* -ALGEBRAS

So far, we have only studied closed operator convex cones rather abstractly. In order to construct actual examples, we use the notions of actions of topological spaces on C^* -algebras. Such actions have been studied quite extensively. In particular, continuous actions of finite spaces have been well studied, with respect to the classification of separable, nuclear, purely infinite C^* -algebras with finite primitive ideal spaces, using K -theoretic invariants.

5.1. The basics. When \mathfrak{A} is a C^* -algebra we let $\mathbb{I}(\mathfrak{A})$ denote the complete lattice of two-sided, closed ideals in \mathfrak{A} .

If X is a topological space we let $\mathbb{O}(X)$ denote the complete lattice of open subsets of X . For a subset Y of X we let Y° denote the interior of Y . Recall, that if $\text{Prim } \mathfrak{A}$ is the primitive ideal space of \mathfrak{A} then there is a canonical complete lattice isomorphism $\mathbb{O}(\text{Prim } \mathfrak{A}) \cong \mathbb{I}(\mathfrak{A})$.

Definition 5.1. Let X be a topological space. An *action of X on a C^* -algebra \mathfrak{A}* is an order preserving map

$$\mathbb{O}(X) \rightarrow \mathbb{I}(\mathfrak{A}), \quad U \mapsto \mathfrak{A}(U),$$

i.e. a map such that if $U \subseteq V$ in $\mathbb{O}(X)$ then $\mathfrak{A}(U) \subseteq \mathfrak{A}(V)$.

If \mathfrak{A} is a C^* -algebra with an action of X , then we say that \mathfrak{A} (together with this action) is an X - C^* -algebra. We will almost always suppress the action in the notation.

A map $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ of X - C^* -algebras is called X -equivariant if $\phi(\mathfrak{A}(U)) \subseteq \mathfrak{B}(U)$ for every $U \in \mathcal{O}(X)$.

An *isomorphism of X - C^* -algebras* is an X -equivariant $*$ -isomorphism $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ such that the inverse ϕ^{-1} is also X -equivariant.

Remark 5.2. If \mathfrak{A} and \mathfrak{B} are X - C^* -algebras, then the set $CP(X; \mathfrak{A}, \mathfrak{B})$ of X -equivariant c.p. maps is a closed operator convex cone.

Remark 5.3. As explained in [MN09, Section 2.5], for any topological space X , we may find an induced *sober* topological space \hat{X} such that there is an induced complete lattice isomorphism $\mathcal{O}(X) \cong \mathcal{O}(\hat{X})$. If X is already sober then $X = \hat{X}$. In particular, we may always assume that X is a sober space.

Since any sober space is a T_0 -space (i.e. for any two distinct points in X , one of the points contains an open neighbourhood not containing the other point), we may assume that X is a T_0 -space.

It is often necessary to impose stronger conditions on our actions.

Definition 5.4. Let X be a topological space and \mathfrak{A} be an X - C^* -algebra. We say that \mathfrak{A} is

- *finitely lower semicontinuous* if $\mathfrak{A}(X) = \mathfrak{A}$, and if it respects finite infima, i.e. for open subsets U and V of X we have

$$\mathfrak{A}(U) \cap \mathfrak{A}(V) = \mathfrak{A}(U \cap V),$$

- *lower semicontinuous* if $\mathfrak{A}(X) = \mathfrak{A}$, and if it respects arbitrary infima, i.e. for any family (U_α) of open subsets of X we have

$$\bigcap_{\alpha} \mathfrak{A}(U_\alpha) = \mathfrak{A}(U),$$

where U is the interior of $\bigcap_{\alpha} U_\alpha$,

- *finitely upper semicontinuous* if $\mathfrak{A}(\emptyset) = 0$ and if it respects finite suprema, i.e. for open subsets U and V of X we have

$$\mathfrak{A}(U) + \mathfrak{A}(V) = \mathfrak{A}(U \cup V),$$

- *monotone upper semicontinuous* if it respects monotone suprema, i.e. for any increasing net (U_α) of open subsets of X we have

$$\overline{\bigcup_{\alpha} \mathfrak{A}(U_\alpha)} = \mathfrak{A}\left(\bigcup_{\alpha} U_\alpha\right),$$

- *upper semicontinuous* if it is finitely and monotone upper semicontinuous,
- *finitely continuous* if it is finitely lower semicontinuous and finitely upper semicontinuous,
- *continuous* if it is lower semicontinuous and upper semicontinuous,
- *tight* if the action $\mathcal{O}(X) \rightarrow \mathbb{I}(\mathfrak{A})$ is a complete lattice isomorphism.

Remark 5.5. Note that we require (finitely) lower semicontinuous X - C^* -algebras \mathfrak{A} to satisfy $\mathfrak{A}(X) = \mathfrak{A}$, and (finitely) upper semicontinuous X - C^* -algebras \mathfrak{B} to satisfy $\mathfrak{B}(\emptyset) = 0$.

Although we will not use it in this paper, the reason that we require this in the upper semicontinuous case, is so that the map

$$\mathbb{I}(\mathfrak{A}) \rightarrow \mathbb{O}(X), \quad \mathfrak{J} \mapsto \bigcup \{U \in \mathbb{O}(X) : \mathfrak{A}(U) \subset \mathfrak{J}\}$$

is well-defined. The reason in the finitely upper semicontinuous case is basically the same, since the action of any finite filtration (which we define in Section 5.4) will be upper semicontinuous.

Example 5.6. If X is a locally compact Hausdorff space, then a continuous X - C^* -algebra is essentially the same as a continuous $C_0(X)$ -algebra. In fact, the action of a continuous X - C^* -algebra induces a non-degenerate $*$ -homomorphism from $C_0(X)$ to the centre of $\mathcal{M}(\mathfrak{B})$ which defines a continuous $C_0(X)$ -algebra structure, and vice versa. This is shown in [MN09, Section 2.2]. It is also shown that a $*$ -homomorphism is X -equivariant if and only if it is $C(X)$ -linear. The same proof applies to c.p. maps, thus a c.p. map is X -equivariant if and only if it is $C(X)$ -linear.

Remark 5.7 (X - C^* -algebras versus C^* -algebras over X). A C^* -algebra over X is a C^* -algebra \mathfrak{A} together with a continuous map $\text{Prim } \mathfrak{A} \rightarrow X$. It is shown in [MN09] that a C^* -algebra over X is essentially the same thing as a finitely lower semicontinuous, upper semicontinuous X - C^* -algebra. Thus we abuse notation slightly, by saying that a C^* -algebra over X is a finitely lower semicontinuous, upper semicontinuous X - C^* -algebra.

In [MN09, Section 2.9] Meyer and Nest give good reasons for why they only want to consider C^* -algebras over X . However, only considering C^* -algebras over X is a great limitation on the constructions we allow ourselves to do, and seems to be more of a restraint than a simplification.

As an example, let us consider one of the most elementary construction for C^* -algebras; the unitisation. If \mathfrak{A} is a C^* -algebra over X , there is no natural way to give the forced unitisation \mathfrak{A}^\dagger a C^* -algebra over X structure. However, if \mathfrak{A} is an X - C^* -algebra then we may give the forced unitisation \mathfrak{A}^\dagger an X - C^* -algebra structure by letting $\mathfrak{A}^\dagger(U) = \mathfrak{A}(U)$ for $U \neq X$ and $\mathfrak{A}^\dagger(X) = \mathfrak{A}^\dagger$. One can easily verify that for X - C^* -algebras \mathfrak{A} and \mathfrak{B} we have

$$CP(X; \mathfrak{A}^\dagger, \mathfrak{B}) = CP(X; \mathfrak{A}, \mathfrak{B}(X))^\dagger,$$

where we used the notation from Lemma 3.9.

The forced unitisation always preserves lower semicontinuity and finite lower semicontinuity. However, it preserves monotone upper semicontinuity if and only if X is compact, and it preserves finite upper semicontinuity if and only if X is *not* the union of two proper open subsets. Thus, for most spaces X , the forced unitisation of a C^* -algebra over X is not a C^* -algebra over X .

5.2. Lower semicontinuous X - C^* -algebras and full maps.

Definition 5.8. Let \mathfrak{A} be an X - C^* -algebra, $a \in \mathfrak{A}$ and $U \in \mathbb{O}(X)$. We say that a is U -full, if U is the unique smallest open subset of X such that $a \in \mathfrak{A}(U)$, i.e. $a \in \mathfrak{A}(U)$ and whenever $V \in \mathbb{O}(X)$ such that $a \in \mathfrak{A}(V)$ then $U \subseteq V$.

It turns out that every element in an X - C^* -algebra being U -full for some U , is equivalent to the X - C^* -algebra being lower semicontinuous.

Proposition 5.9. *Let \mathfrak{A} be an X - C^* -algebra. Then \mathfrak{A} is lower semicontinuous if and only if every element $a \in \mathfrak{A}$ is U -full for some (unique) $U \in \mathcal{O}(X)$.*

Proof. Suppose that \mathfrak{A} is lower semicontinuous and let $a \in \mathfrak{A}$. Let $\mathcal{U} = \{V \in \mathcal{O}(X) : a \in \mathfrak{A}(V)\}$, which is non-empty since $\mathfrak{A}(X) = \mathfrak{A}$, and define

$$U = \left(\bigcap_{V \in \mathcal{U}} V \right)^\circ.$$

By lower semicontinuity $\bigcap_{V \in \mathcal{U}} \mathfrak{A}(V) = \mathfrak{A}(U)$ which contains a . Hence by construction a is U -full.

Suppose that every element in \mathfrak{A} is U -full for some U , and let (U_α) be a family of open subsets of X . Define $U_0 := (\bigcap_\alpha U_\alpha)^\circ$ and note that we obviously have an inclusion $\mathfrak{A}(U_0) \subseteq \bigcap \mathfrak{A}(U_\alpha)$. Let $a \in \bigcap \mathfrak{A}(U_\alpha)$ and $U \in \mathcal{O}(X)$ such that a is U -full. Since $a \in \mathfrak{A}(U_\alpha)$ for all α , $U \subseteq U_\alpha$ for each α . Hence $U \subseteq U_0$, implying that $a \in \mathfrak{A}(U_0)$. Thus $\mathfrak{A}(U_0) = \bigcap \mathfrak{A}(U_\alpha)$. Finally, if $a \in \mathfrak{A} \setminus \mathfrak{A}(X)$ and a is U -full for some $U \in \mathcal{O}(X)$, then $a \in \mathfrak{A}(U) \subset \mathfrak{A}(X)$ which is a contradiction. Thus $\mathfrak{A}(X) = \mathfrak{A}$ and thus \mathfrak{A} is lower semicontinuous. \square

Notation 5.10. If \mathfrak{A} is a lower semicontinuous X - C^* -algebra, and $a \in \mathfrak{A}$, then we denote by U_a the unique open subset of X for which a is U_a -full.

Example 5.11. Let X be a locally compact Hausdorff space and \mathfrak{A} be a continuous $C_0(X)$ -algebra. For $a \in \mathfrak{A}$ it is easily seen that

$$U_a = \{x \in X : \|a_x\|_x > 0\}.$$

where a_x denotes the fibre at x .

Observation 5.12. Let \mathfrak{A} and \mathfrak{B} be X - C^* -algebras with \mathfrak{A} lower semicontinuous. Then a map $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ is X -equivariant if and only if for all $a \in \mathfrak{A}$, $\phi(a) \in \mathfrak{B}(U_a)$.

Recall, that an element b in a C^* -algebra \mathfrak{B} is called *full* if the closed, two-sided ideal generated by b is all of \mathfrak{B} .

Definition 5.13 (Full X -equivariant maps). Let \mathfrak{A} and \mathfrak{B} be X - C^* -algebras with \mathfrak{A} lower semicontinuous, and let $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ be an X -equivariant map. We say that ϕ is *full* (or X -full) if $\phi(a)$ is full in $\mathfrak{B}(U_a)$ for all $a \in \mathfrak{A}$.

Note that simply the existence of an X -full map $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ puts a restraint on the action on \mathfrak{B} . In fact, we always have that $0 \in \mathfrak{A}$ is \emptyset -full, and thus $\phi(0) = 0$ must be full in $\mathfrak{B}(\emptyset)$. Thus $\mathfrak{B}(\emptyset) = 0$. However, this is a very small requirement, and since in all our applications \mathfrak{B} will be finitely upper semicontinuous, this obstruction will never be an issue.

Example 5.14. Let $X = \star$ be the one-point space. Every C^* -algebra is canonically a continuous X - C^* -algebra, and any linear map $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ is canonically X -equivariant. Clearly any non-zero element $a \in \mathfrak{A}$ is X -full, and one always has that 0 is \emptyset -full. Note that $\phi(0) = 0$ is full in $\mathfrak{B}(\emptyset) = 0$. Hence ϕ is full in the X -equivariant sense exactly when $\phi(a)$ is full in \mathfrak{B} for every non-zero $a \in \mathfrak{A}$. This is exactly the classical definition of a linear map being full.

One of the applications is that if there exists a full map we get a description of the map $\mathfrak{B}_e: \mathfrak{A} \rightarrow \mathbb{I}(\mathfrak{B})$ c.f. Question 4.15.

Lemma 5.15. *Let \mathfrak{A} and \mathfrak{B} be X - C^* -algebras with \mathfrak{A} lower semicontinuous, and let $\mathcal{C} \subset CP(X; \mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Let $a \in \mathfrak{A}$. If there is $\phi \in \mathcal{C}$ such that $\phi(a)$ is full in $\mathfrak{B}(U_a)$, then $\mathfrak{B}_{\mathcal{C}}(a) = \mathfrak{B}(U_a)$.*

In particular, if \mathcal{C} contains an X -full map, then $\mathfrak{B}_{\mathcal{C}}(a) = \mathfrak{B}(U_a)$ for every $a \in \mathfrak{A}$.

Proof. Since every map in \mathcal{C} is X -equivariant, we clearly have $\mathfrak{B}_{\mathcal{C}}(a) \subset \mathfrak{B}(U_a)$ for all $a \in \mathfrak{A}$. Let $a \in \mathfrak{A}$ for which there exists $\phi \in \mathcal{C}$ such that $\phi(a)$ is full in $\mathfrak{B}(U_a)$. Then $\mathfrak{B}(U_a) = \overline{\mathfrak{B}\phi(a)\mathfrak{B}} \subset \mathfrak{B}_{\mathcal{C}}(a)$. Thus $\mathfrak{B}_{\mathcal{C}}(a) = \mathfrak{B}(U_a)$. The last part of the lemma follows immediately since if \mathcal{C} contains an X -full map ϕ , then for every $a \in \mathfrak{A}$, we have that $\phi(a)$ is full in $\mathfrak{B}(U_a)$. \square

5.3. Basic X -equivariant extension theory. Given a two-sided, closed ideal \mathfrak{J} in a C^* -algebra \mathfrak{B} there are canonical induced two-sided, closed ideals in the multiplier algebra $\mathcal{M}(\mathfrak{B})$ and the corona algebra $\mathcal{Q}(\mathfrak{B})$, given by

$$\begin{aligned} \mathcal{M}(\mathfrak{B}, \mathfrak{J}) &= \{m \in \mathcal{M}(\mathfrak{B}) : m\mathfrak{B} \subseteq \mathfrak{J}\}, \\ \mathcal{Q}(\mathfrak{B}, \mathfrak{J}) &= (\mathcal{M}(\mathfrak{B}, \mathfrak{J}) + \mathfrak{B})/\mathfrak{B}. \end{aligned}$$

Note that $\mathcal{M}(\mathfrak{B}, \mathfrak{J}) \cap \mathfrak{B} = \mathfrak{J}$.

Given an action of a space X on a C^* -algebra \mathfrak{B} , there are induced actions on $\mathcal{M}(\mathfrak{B})$ and $\mathcal{Q}(\mathfrak{B})$ given by $\mathcal{M}(\mathfrak{B})(U) = \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$ and $\mathcal{Q}(\mathfrak{B})(U) = \mathcal{Q}(\mathfrak{B}, \mathfrak{B}(U))$.

Remark 5.16. The embedding $\mathcal{M}(\mathfrak{B})(U) \hookrightarrow \mathcal{M}(\mathfrak{B})(U) + \mathfrak{B}$ induces an isomorphism

$$\frac{\mathcal{M}(\mathfrak{B})(U)}{\mathfrak{B}(U)} = \frac{\mathcal{M}(\mathfrak{B})(U)}{\mathcal{M}(\mathfrak{B})(U) \cap \mathfrak{B}} \xrightarrow{\cong} \frac{\mathcal{M}(\mathfrak{B})(U) + \mathfrak{B}}{\mathfrak{B}} = \mathcal{Q}(\mathfrak{B})(U),$$

for every $U \in \mathbb{O}(X)$. This implies that any element $\mathcal{Q}(\mathfrak{B})(U)$ lifts to an element in $\mathcal{M}(\mathfrak{B})(U) \subset \mathcal{M}(\mathfrak{B})$. Also, for any $U \in \mathbb{O}(X)$ this implies that the sequence $0 \rightarrow \mathfrak{B}(U) \rightarrow \mathcal{M}(\mathfrak{B})(U) \rightarrow \mathcal{Q}(\mathfrak{B})(U) \rightarrow 0$ induced by $0 \rightarrow \mathfrak{B} \rightarrow \mathcal{M}(\mathfrak{B}) \rightarrow \mathcal{Q}(\mathfrak{B}) \rightarrow 0$ is a short exact sequence.

We have the following.

Lemma 5.17. *Let \mathfrak{B} be an X - C^* -algebra.*

- (a) *If \mathfrak{B} is finitely lower semicontinuous, then $\mathcal{M}(\mathfrak{B})$ and $\mathcal{Q}(\mathfrak{B})$ are finitely lower semicontinuous.*
- (b) *If \mathfrak{B} is lower semicontinuous, then $\mathcal{M}(\mathfrak{B})$ is lower semicontinuous.*
- (c) *If \mathfrak{B} is σ -unital and finitely upper semicontinuous, then $\mathcal{M}(\mathfrak{B})$ and $\mathcal{Q}(\mathfrak{B})$ are finitely upper semicontinuous.*

Proof. Part (b) is obvious, and part (a) for the multiplier algebra is also clear. Thus it suffices to show that $\mathcal{Q}(\mathfrak{B})$ is finitely lower semicontinuous whenever \mathfrak{B} is, where we obviously have $\mathcal{Q}(\mathfrak{B})(X) = \mathcal{Q}(\mathfrak{B})$.

Let $U, V \in \mathbb{O}(X)$. Clearly $\mathcal{Q}(\mathfrak{B})(U \cap V) \subset \mathcal{Q}(\mathfrak{B})(U) \cap \mathcal{Q}(\mathfrak{B})(V)$. It suffices to show that the image of this inclusion is dense. Fix $x \in \mathcal{Q}(\mathfrak{B})(U) \cap \mathcal{Q}(\mathfrak{B})(V)$. By Remark 5.16 we may lift x to an element $y \in \mathcal{M}(\mathfrak{B})(U)$. Let (z_α) be an approximate identity of $\mathcal{M}(\mathfrak{B})(V)$. Then yz_α is in $\mathcal{M}(\mathfrak{B})(U \cap V)$ and thus $\pi(yz_\alpha)$ is in $\mathcal{Q}(\mathfrak{B})(U \cap V)$ and converges to x .

Ad (c): Clearly $\mathcal{M}(\mathfrak{B})(\emptyset) = 0$ and $\mathcal{Q}(\mathfrak{B})(\emptyset) = 0$.

We start by showing that $\mathcal{M}(\mathfrak{B})$ is finitely upper semicontinuous and use this to show that $\mathcal{Q}(\mathfrak{B})$ is also finitely upper semicontinuous. Let $U, V \in \mathbb{O}(X)$. Clearly $\mathcal{M}(\mathfrak{B})(U) +$

$\mathcal{M}(\mathfrak{B})(V) \subset \mathcal{M}(\mathfrak{B})(U \cup V)$. Let $m \in \mathcal{M}(\mathfrak{B})(U \cup V)$ be a positive contraction and let (b_n) be a countable approximate identity in \mathfrak{B} such that $b_{n+1}b_n = b_nb_{n+1} = b_n$. By passing to a subsequence we may assume that

$$\|(1 - b_n)mb_k\| < 2^{-(n+k)},$$

for all $k < n$. Let $d_n = b_n - b_{n-1}$ where we define $b_0 = 0$.

We claim that by using the norm estimate above one gets that m is in the two-sided, closed ideal

$$\overline{\mathcal{M}(\mathfrak{B})\left(\sum_{n=1}^{\infty} d_n m d_n\right)\mathcal{M}(\mathfrak{B}) + \mathfrak{B}(U \cup V)}.$$

To see this note first that $\sum_{n=1}^{\infty} d_n = 1$ strictly. Thus

$$\begin{aligned} m &= \sum_{k,l=1}^{\infty} d_k m d_l \\ &= \sum_{n=1}^{\infty} d_n m d_n + \sum_{n=1}^{\infty} (d_{n+1} m d_n + d_n m d_{n+1}) + \sum_{l=3}^{\infty} \sum_{k=1}^{l-2} (d_k m d_l + d_l m d_k). \end{aligned}$$

The latter of these three sums is in $\mathfrak{B}(U \cup V)$ since $\|d_l m d_k\| = \|b_l(1 - b_{l-1})m d_k\| < 2^{-(k+l)}$. Thus it suffices to show that the second sum is in the two-sided, closed ideal generated by the first sum plus something from $\mathfrak{B}(U \cup V)$. Using that $d_n d_l = 0$ for $n < l - 1$, we get that

$$\begin{aligned} &\left(\sum_{n=1}^{\infty} d_{n+1} m d_n\right)^* \left(\sum_{n=1}^{\infty} d_{n+1} m d_n\right) \\ &= \sum_{n=1}^{\infty} d_n m d_{n+1}^2 m d_n + \sum_{n=1}^{\infty} (d_n m d_{n+1} d_{n+2} m d_{n+1} + d_{n+1} m d_{n+2} d_{n+1} m d_n) \end{aligned}$$

Using that $d_{n+1}d_{n+2} = d_{n+2}d_{n+1}$, the same norm consideration as above implies that the second of these sums is in $\mathfrak{B}(U \cup V)$. Finally, noting that

$$\sum_{n=1}^{\infty} d_n m d_{n+1}^2 m d_n \leq \sum_{n=1}^{\infty} d_n m^2 d_n \leq \sum_{n=1}^{\infty} d_n m d_n,$$

the claim follows.

By the claim, it suffices to show that $\sum_{n=1}^{\infty} d_n m d_n$ is in $\mathcal{M}(\mathfrak{B})(U) + \mathcal{M}(\mathfrak{B})(V)$. For each n we may find positive contractions $x_n \in \mathfrak{B}(U)$, $y_n \in \mathfrak{B}(V)$ such that $x_n + y_n = d_n^{1/2} m d_n^{1/2}$. Then

$$\sum_{n=1}^{\infty} d_n m d_n = \sum_{n=1}^{\infty} d_n^{1/2} x_n d_n^{1/2} + \sum_{n=1}^{\infty} d_n^{1/2} y_n d_n^{1/2} \in \mathcal{M}(\mathfrak{B})(U) + \mathcal{M}(\mathfrak{B})(V).$$

Hence $m \in \mathcal{M}(\mathfrak{B})(U) + \mathcal{M}(\mathfrak{B})(V)$, so $\mathcal{M}(\mathfrak{B})$ is finitely upper semicontinuous.

Now for the corona algebra. Clearly $\mathcal{Q}(\mathfrak{B})(U) + \mathcal{Q}(\mathfrak{B})(V) \subset \mathcal{Q}(\mathfrak{B})(U \cup V)$. By Remark 5.16 any element in $\mathcal{Q}(\mathfrak{B})(U \cup V)$ lifts to one in $\mathcal{M}(\mathfrak{B})(U \cup V)$, which is $\mathcal{M}(\mathfrak{B})(U) + \mathcal{M}(\mathfrak{B})(V)$ by what we have already proven. Hence it follows that $\mathcal{Q}(\mathfrak{B})$ is upper semicontinuous. \square

Example 5.18. If \mathfrak{B} is a lower semicontinuous X - C^* -algebra, then $\mathcal{Q}(\mathfrak{B})$ is not necessarily lower semicontinuous. As an example, take $\mathfrak{B} = C_0((0, 1])$ as a tight $(0, 1]$ - C^* -algebra. Recall that $\mathcal{M}(\mathfrak{B})$ may naturally be identified with $C_b((0, 1])$, the C^* -algebra of bounded continuous functions on $(0, 1]$. Then $\mathcal{Q}(\mathfrak{B})((0, 1/n)) = \mathcal{Q}(\mathfrak{B})$ for all n . However, since $\bigcap_{n \in \mathbb{N}} (0, 1/n)$ is empty, if $\mathcal{Q}(\mathfrak{B})$ was lower semicontinuous then $\bigcap \mathcal{Q}(\mathfrak{B})((0, 1/n))$ should be 0, which it is not.

Similarly, if \mathfrak{B} is a monotone upper semicontinuous X - C^* -algebra, then $\mathcal{M}(\mathfrak{B})$ and $\mathcal{Q}(\mathfrak{B})$ are not necessarily monotone upper semicontinuous. To see this let \mathfrak{B} be as above. Then $\mathcal{M}(\mathfrak{B})((1/n, 1]) = \mathfrak{B}((1/n, 1])$. Hence $\bigcup_{n=1}^{\infty} \mathcal{M}(\mathfrak{B})((1/n, 1]) = \mathfrak{B} \neq \mathcal{M}(\mathfrak{B})$, even though $\bigcup (1/n, 1] = (0, 1]$. This also works for the corona algebra, since $\mathcal{Q}(\mathfrak{B})((1/n, 1]) = 0$, but $\mathcal{Q}(\mathfrak{B})((0, 1]) = \mathcal{Q}(\mathfrak{B})$.

Definition 5.19. An extension $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ of C^* -algebras, for which \mathfrak{A} , \mathfrak{B} and \mathfrak{E} are X - C^* -algebras is called *X -equivariant* if the $*$ -homomorphisms are X -equivariant and the induced sequence $0 \rightarrow \mathfrak{B}(\mathsf{U}) \rightarrow \mathfrak{E}(\mathsf{U}) \rightarrow \mathfrak{A}(\mathsf{U}) \rightarrow 0$ is exact for each $\mathsf{U} \in \mathbb{O}(\mathsf{X})$.

When such an extension is X -equivariant, we will say that it is an *extension of X - C^* -algebras*.

An example of this was given in Remark 5.16. Here we saw, that if \mathfrak{B} is an X - C^* -algebra, and $\mathcal{M}(\mathfrak{B})$ and $\mathcal{Q}(\mathfrak{B})$ are given the induced X - C^* -algebra structures, then the extension $0 \rightarrow \mathfrak{B} \rightarrow \mathcal{M}(\mathfrak{B}) \rightarrow \mathcal{Q}(\mathfrak{B}) \rightarrow 0$ is X -equivariant.

If a Busby map $\tau: \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$ is X -equivariant, then the pull-back $\mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B}) = \{(a, m) \in \mathfrak{A} \oplus \mathcal{M}(\mathfrak{B}) : \tau(a) = \pi(m)\}$ gets an induced action of X by

$$(\mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B}))(\mathsf{U}) = (\mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B})) \cap (\mathfrak{A}(\mathsf{U}) \oplus \mathcal{M}(\mathfrak{B})(\mathsf{U}))$$

for $\mathsf{U} \in \mathbb{O}(\mathsf{X})$.

Proposition 5.20. Let $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras such that \mathfrak{A} , \mathfrak{B} and \mathfrak{E} are X - C^* -algebras and the $*$ -homomorphisms are X -equivariant. The following are equivalent.

- (i) The extension is X -equivariant,
- (ii) $\mathfrak{B}(\mathsf{U}) = \mathfrak{B} \cap \mathfrak{E}(\mathsf{U})$ and $\mathfrak{A}(\mathsf{U}) = p(\mathfrak{E}(\mathsf{U}))$ for all $\mathsf{U} \in \mathbb{O}(\mathsf{X})$,
- (iii) The Busby map τ is X -equivariant, and the induced isomorphism

$$\mathfrak{E} \xrightarrow{\cong} \mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B})$$

is an isomorphism of X - C^* -algebras.

Proof. (i) \Rightarrow (ii): If the extension is X -equivariant then clearly the map p induces an isomorphism $(\mathfrak{E}(\mathsf{U}) + \mathfrak{B})/\mathfrak{B} \xrightarrow{\cong} \mathfrak{A}(\mathsf{U})$. The inclusion $\mathfrak{E}(\mathsf{U}) \hookrightarrow \mathfrak{E}(\mathsf{U}) + \mathfrak{B}$ induces an isomorphism such that $p|_{\mathfrak{E}(\mathsf{U})}: \mathfrak{E}(\mathsf{U}) \rightarrow \mathfrak{A}(\mathsf{U})$ is the composition

$$\mathfrak{E}(\mathsf{U}) \twoheadrightarrow \frac{\mathfrak{E}(\mathsf{U})}{\mathfrak{E}(\mathsf{U}) \cap \mathfrak{B}} \xrightarrow{\cong} \frac{\mathfrak{E}(\mathsf{U}) + \mathfrak{B}}{\mathfrak{B}} \xrightarrow{\cong} \mathfrak{A}(\mathsf{U}).$$

It follows from exactness that $\mathfrak{B}(\mathsf{U}) = \ker p|_{\mathfrak{E}(\mathsf{U})} = \mathfrak{E}(\mathsf{U}) \cap \mathfrak{B}$.

(ii) \Rightarrow (i): Since $p: \mathfrak{E}(\mathsf{U}) \rightarrow \mathfrak{A}(\mathsf{U})$ is surjective by assumption, it suffices to show that the kernel is $\mathfrak{B}(\mathsf{U})$. As above, the kernel is $\mathfrak{E}(\mathsf{U}) \cap \mathfrak{B} = \mathfrak{B}(\mathsf{U})$.

(ii) \Rightarrow (iii): Let $\sigma: \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ be the canonical $*$ -homomorphism. Note that

$$\mathfrak{E}(U) \cdot \mathfrak{B} = \mathfrak{B}(U).$$

Thus for any $x \in \mathfrak{E}(U)$, we have $x\mathfrak{B} \subset \mathfrak{B}(U)$, which implies that $\sigma(x) \in \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U)) = \mathcal{M}(\mathfrak{B})(U)$. Hence σ is X -equivariant. Recall that τ is constructed as follows: let $a \in \mathfrak{A}$, lift it to any element $x \in \mathfrak{E}$, and then let $\tau(a) = \pi \circ \sigma(a)$. If $a \in \mathfrak{A}(U)$, then by (ii) we may lift a to an element $x \in \mathfrak{E}(U)$. Since σ and π are both X -equivariant, $\tau(a) = \pi \circ \sigma(x) \in \mathcal{Q}(\mathfrak{B})(U)$. Thus τ is X -equivariant. Since $\mathfrak{E} \xrightarrow{\cong} \mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B})$ is induced by the X -equivariant maps p and σ , it follows that this isomorphism is X -equivariant.

It remains to show that the inverse is also X -equivariant. Suppose $(a, m) \in (\mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B}))(U)$, and let $x \in \mathfrak{E}$ be such that $p(x) = a$ and $\sigma(x) = m$. We must show that $x \in \mathfrak{E}(U)$. Lift a to an element $y \in \mathfrak{E}(U)$. Then $b := \sigma(x - y) = m - \sigma(y) \in \mathfrak{B} \cap \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U)) = \mathfrak{B}(U)$. Since $\mathfrak{B}(U) \subset \mathfrak{E}(U)$ we have that $b + y \in \mathfrak{E}(U)$, and satisfies $p(b + y) = a$ and $\sigma(b + y) = m$. Thus $x = b + y \in \mathfrak{E}(U)$.

(iii) \Rightarrow (ii): Suppose that τ is X -equivariant, and that when identifying \mathfrak{E} with the pull-back $\mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B})$, then

$$\mathfrak{E}(U) = (\mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B})) \cap (\mathfrak{A}(U) \oplus \mathcal{M}(\mathfrak{B})(U))$$

for all $U \in \mathbb{O}(X)$. We have $\mathfrak{B}(U) = \mathfrak{B} \cap \mathcal{M}(\mathfrak{B})(U) = \mathfrak{B} \cap \mathfrak{E}(U)$. Clearly $p(\mathfrak{E}(U)) \subseteq \mathfrak{A}(U)$ since p is X -equivariant by assumption. If $a \in \mathfrak{A}(U)$ we may lift $\tau(a) \in \mathcal{Q}(\mathfrak{B})(U)$ to an element $m \in \mathcal{M}(\mathfrak{B})(U)$ by Remark 5.16. Hence $(a, m) \in \mathfrak{E}(U)$ which implies that $p(\mathfrak{E}(U)) = \mathfrak{A}(U)$. \square

Note that it was proven in (ii) \Rightarrow (iii) above, that if $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ is an extension of X - C^* -algebras, then the induced $*$ -homomorphism $\sigma: \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ is X -equivariant.

As an analogue of full extensions of C^* -algebras, we make a similar definition for X -equivariant extensions when the action on the quotient algebra is lower semicontinuous. These will play an important part later, especially in Theorem 6.22 and its corollaries.

Definition 5.21. Let $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of X - C^* -algebras, such that \mathfrak{A} is lower semicontinuous. We say that the extension is *full* (or X -full) if the Busby map is X -full.

5.4. An approximation property for residually X -nuclear maps. It follows from Theorem 3.12, that in order to show that a closed operator convex cone \mathcal{C} satisfies the purely large problem, it suffices to find a generating subset \mathcal{S} consisting of suitably well-behaved maps. E.g. in the classical case where $\mathcal{C} = CP_{\text{nuc}}(\mathfrak{A}, \mathfrak{B})$, one uses that these maps have the classical approximation property of nuclear maps, i.e. that the maps approximately factor through matrix algebras.

In order to solve our purely large problem for X - C^* -algebras with X finite, we will study residually X -nuclear maps, which we do for not necessarily finite spaces X . In fact, we prove an approximation property for all such maps, which can also be used to prove other nice results, such as a Choi–Effros type lifting theorem for residually X -nuclear maps, when X is finite.

Notation 5.22. Whenever $\Psi: \mathfrak{A} \rightarrow \mathfrak{B}$ is an X -equivariant linear map and $U \in \mathbb{O}(X)$ we let $[\Psi]_U: \mathfrak{A}/\mathfrak{A}(U) \rightarrow \mathfrak{B}/\mathfrak{B}(U)$ be the induced map.

Definition 5.23. Let \mathfrak{A} and \mathfrak{B} be X - C^* -algebras, and let $\Psi: \mathfrak{A} \rightarrow \mathfrak{B}$ be an X -equivariant nuclear map. We say that Ψ is *residually X -nuclear* if $[\Psi]_U$ is nuclear for any $U \in \mathcal{O}(X)$.

We let $CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})$ denote the set of residually X -nuclear maps.

Remark 5.24. It is straight forward to verify that $CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})$ is a closed operator convex cone.

Remark 5.25. Note that if \mathfrak{A} or \mathfrak{B} is nuclear, then any X -equivariant c.p. map is residually X -nuclear. Moreover, if \mathfrak{A} is exact (or even locally reflexive), then an X -equivariant c.p. map $\mathfrak{A} \rightarrow \mathfrak{B}$ is residually X -nuclear exactly when it is (non-equivariantly) nuclear, i.e.

$$CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B}) = CP(X; \mathfrak{A}, \mathfrak{B}) \cap CP_{\text{nuc}}(\mathfrak{A}, \mathfrak{B}).$$

This follows from [Dad97, Proposition 3.2].

Notation 5.26. Let X be a topological space and Y be a finite space. Then any surjective continuous map $f: X \rightarrow Y$ is called a *finite filtration* of X . We will often just say that Y is a finite filtration of X .

Given a finite filtration Y of X , then any X - C^* -algebra \mathfrak{A} gets an induced Y - C^* -algebra structure, by letting $\mathfrak{A}(U) = \mathfrak{A}(f^{-1}(U))$ for any X - C^* -algebra \mathfrak{A} and any $U \in \mathcal{O}(Y)$.

When given a topological space X , a C^* -algebra \mathfrak{A} , and a non-empty subset $A \subset X$, we may give \mathfrak{A} an induced X - C^* -algebra structure by letting $\mathfrak{A}(U) = \mathfrak{A}$ if $A \subseteq U$ and 0 if $A \not\subseteq U$. We will let $i_A(\mathfrak{A})$ denote \mathfrak{A} with this X - C^* -algebra structure.

Lemma 5.27. Let Y be a finite T_0 space and \mathfrak{D} is a finite dimensional, continuous Y - C^* -algebra. Then $\mathfrak{D} \cong \bigoplus_{y \in Y} i_{\{y\}}(\mathfrak{D}_y)$ as Y - C^* -algebras for some (necessarily finite dimensional) C^* -algebras \mathfrak{D}_y .

Proof. Let U^y denote the smallest open subset of Y which contains y . Note that $U^y \setminus \{y\}$ is also open. Let $\mathfrak{D}_y = \mathfrak{D}(U^y) / \mathfrak{D}(U^y \setminus \{y\})$. Since \mathfrak{D} is finite dimensional, any two-sided, closed ideal is a direct summand. Thus there is a canonical induced Y -equivariant injective $*$ -homomorphism $\iota_y: i_{\{y\}}(\mathfrak{D}_y) \rightarrow \mathfrak{D}$. Since \mathfrak{D} is lower semicontinuous and $\mathfrak{D}(\emptyset) = 0$, it follows that all ι_y have orthogonal images. Thus these induce a Y -equivariant injective $*$ -homomorphism $\iota: \bigoplus_{y \in Y} i_{\{y\}}(\mathfrak{D}_y) \rightarrow \mathfrak{D}$. Since \mathfrak{D} is upper semicontinuous, $\mathfrak{D}(U) = \sum_{y \in U} \mathfrak{D}(U^y)$ which is canonically isomorphic to $\bigoplus_{y \in U} \mathfrak{D}_y$ for any $U \in \mathcal{O}(Y)$. Thus it easily follows that ι induces an isomorphism of Y - C^* -algebras. \square

Lemma 5.28. Let Y be a finite space, \mathfrak{D} be a finite dimensional Y - C^* -algebra, and $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{B} \xrightarrow{p} \mathfrak{B}/\mathfrak{J} \rightarrow 0$ be an extension of Y - C^* -algebras. Suppose that $\rho: \mathfrak{D} \rightarrow \mathfrak{B}/\mathfrak{J}$ is a Y -equivariant contractive c.p. map. Then there exists a Y -equivariant contractive c.p. map $\tilde{\rho}: \mathfrak{D} \rightarrow \mathfrak{B}$ such that $p \circ \tilde{\rho} = \rho$.

Proof. By [MN09, Sections 2.5 and 2.9] we may assume (by replacing Y with a possibly bigger, but still finite, space) that Y is T_0 and that all given Y - C^* -algebras are continuous (i.e. are C^* -algebras over Y since Y is finite). Write $\mathfrak{D} = \bigoplus_{y \in Y} i_{\{y\}}(\mathfrak{D}_y)$ with \mathfrak{D}_y finite dimensional C^* -algebras. Let U^y be the smallest open set containing y . Since ρ is Y -equivariant, the restriction ρ_y to \mathfrak{D}_y factors through $(\mathfrak{B}/\mathfrak{J})(U^y) \cong \mathfrak{B}(U^y) / \mathfrak{J}(U^y)$. Hence this contractive c.p. map lifts to a contractive c.p. map $\tilde{\rho}_y: \mathfrak{D}_y \rightarrow \mathfrak{B}(U^y)$. Let $\hat{\rho}: \mathfrak{D} \rightarrow \mathfrak{B}$ be given by

$$\hat{\rho}(\bigoplus_{y \in Y} d_y) = \sum_{y \in Y} \tilde{\rho}_y(d_y), \quad \text{for } d_y \in \mathfrak{D}_y.$$

This map is clearly Y -equivariant and completely positive but not necessarily contractive. Let $v = f(\hat{\rho}(1_{\mathfrak{D}}))$, defined by functional calculus in the minimal unitisation $\tilde{\mathfrak{B}}$, where $f(t) = \max\{1, t\}$. Clearly $p(v^{-1/2}bv^{-1/2}) = p(b)$ for all $b \in \mathfrak{B}$, and thus the Y -equivariant contractive c.p. map $\tilde{\rho}: \mathfrak{D} \rightarrow \mathfrak{B}$ given by $\tilde{\rho}(d) = v^{-1/2}\hat{\rho}(d)v^{-1/2}$ is also a lift of ρ . \square

The above trick of using the results of [MN09] to assume that our X - C^* -algebras are continuous Y - C^* -algebras, for some larger space Y , will in general not work, since the functors taking X - C^* -algebras to continuous Y - C^* -algebras, will in general *not* take residually X -nuclear maps to residually Y -nuclear maps.

Using the notion of finite filtrations of spaces, we get the following approximately X -equivariant approximation property of residually X -nuclear maps.

Theorem 5.29. *Let \mathfrak{A} and \mathfrak{B} be X - C^* -algebras, with \mathfrak{B} finitely continuous, and let $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ be a c.p. map. Then ϕ is residually X -nuclear if and only if the following holds: for any finite subset $F \subseteq \mathfrak{A}$, any $\epsilon > 0$, and any finite filtration Y of X , there are Y -equivariant c.p. maps*

$$\psi: \mathfrak{A} \rightarrow \mathfrak{D}, \quad \rho: \mathfrak{D} \rightarrow \mathfrak{B},$$

such that

$$\|\phi(a) - \rho(\psi(a))\| < \epsilon, \quad \text{for all } a \in F,$$

where \mathfrak{D} is a finite dimensional Y - C^* -algebra, $\|\psi\| \leq 1$ and $\|\rho\| \leq \|\phi\|$.

Moreover, we may choose \mathfrak{D} to be continuous.

Proof. For the “if” we will start by showing that ϕ is X -equivariant. Let $U \in \mathcal{O}(X)$ and $a \in \mathfrak{A}(U)$ so that we should show that $\phi(a) \in \mathfrak{B}(U)$. If $U = X$, then $\phi(a) \in \mathfrak{B} = \mathfrak{B}(X)$, since \mathfrak{B} is finitely continuous. Thus we assume that $U \neq X$. Suppose $U \neq \emptyset$. Let $Y = \{1, 2\}$ be the two-point space with open subsets $\emptyset, \{2\}, Y$. The map $f: X \rightarrow Y$ given by $f(x) = 2$ if $x \in U$ and $f(x) = 1$ otherwise, is a continuous surjection. Hence $f: X \rightarrow Y$ is a finite filtration. Then choosing any element $a \in \mathfrak{A}(U)$ and any $\epsilon > 0$ we can find maps ρ and ψ as above which are Y -equivariant. Hence $\psi(\rho(a)) \in \mathfrak{B}(\{2\}) = \mathfrak{B}(U)$. Since $\mathfrak{B}(U)$ is closed we get that $\phi(a) \in \mathfrak{B}(U)$. If $U = \emptyset$ let $Y = \{1\}$ and $f: X \rightarrow Y$ be the unique map. The same argument as before shows that $\phi(a) \in \mathfrak{B}(\emptyset) = 0$.

Clearly ϕ is nuclear. In order to prove that ϕ is residually X -nuclear we should show that $[\phi]_U$ is nuclear. Suppose that $U \neq \emptyset$ and let $f: X \rightarrow Y$ be defined as above. Let $F' \subseteq \mathfrak{A}/\mathfrak{A}(U)$ be a finite subset and $\epsilon > 0$. Lift each element a' of F' to an element a of \mathfrak{A} and obtain a finite subset F of \mathfrak{A} . By the approximation property there are Y -equivariant ρ and ψ as described above. We get that

$$\|[\phi]_U(a') - [\psi]_{\{2\}}([\rho]_{\{2\}}(a'))\| \leq \|\phi(a) - \psi(\rho(a))\| < \epsilon, \quad \text{for all } a' \in F'.$$

Hence $[\phi]_U$ is nuclear. If $U = \emptyset$ then a similar argument implies that $[\phi]_U$ is nuclear, and thus ϕ is residually X -nuclear.

Now for the “only if”. We may assume, without loss of generality, that ϕ is contractive. We will prove the result by induction on the number of elements of $\mathcal{O}(Y)$. If $|\mathcal{O}(Y)| = 2$ then $\mathcal{O}(Y) = \{\emptyset, Y\}$. Since \mathfrak{B} is finitely continuous, $\mathfrak{B}(\emptyset) = 0$. Thus $\phi(\mathfrak{A}(\emptyset)) = 0$. Since the map $[\phi]_{\emptyset}$ is nuclear, we may approximate it by maps of the form $\rho \circ \psi$ where $\psi: \mathfrak{A}/\mathfrak{A}(\emptyset) \rightarrow M_n$ and $\rho: M_n \rightarrow \mathfrak{B}$ are contractive by Lemma 2.3. Consider M_n as a continuous Y - C^* -algebra and let $\bar{\psi}$ be the composition $\mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{A}(\emptyset) \xrightarrow{\psi} M_n$. Clearly $\bar{\psi}$ and ρ are Y -equivariant contractive c.p. maps, and $\rho \circ \bar{\psi}$ approximates ϕ . This shows the case $|\mathcal{O}(Y)| = 2$.

Suppose that there is an approximation, as described in the theorem, for any residually Y -nuclear map between Y - C^* -algebras \mathfrak{A} and \mathfrak{B} , with \mathfrak{B} continuous, for any space Y with $|\mathbb{O}(Y)| < n$, where the finite dimensional \mathfrak{D} is a Y - C^* -algebra. Let F, ϵ and Y be given with $|\mathbb{O}(Y)| = n$. Since $\mathbb{O}(Y) \cong \mathbb{O}(\hat{Y})$ for a T_0 -space \hat{Y} by Remark 5.3, we may assume that Y is a T_0 space.

Let $U \in \mathbb{O}(Y)$ be a non-empty, minimal open set (since Y is a finite T_0 space this means that $U = \{y\}$ for some $y \in Y$). Equip $\mathfrak{C}/\mathfrak{C}(U)$ with a $(Y \setminus U)$ - C^* -algebra structure by defining $(\mathfrak{C}/\mathfrak{C}(U))(W) := \mathfrak{C}(W \cup U)/\mathfrak{C}(U)$ for $W \in \mathbb{O}(Y \setminus U)$, and $\mathfrak{C} \in \{\mathfrak{A}, \mathfrak{B}\}$. It is easily seen that $\mathfrak{B}/\mathfrak{B}(U)$ is a continuous $(Y \setminus U)$ - C^* -algebra, and that $[\phi]_U$ is $(Y \setminus U)$ -equivariant. Moreover, for any $W \in \mathbb{O}(Y \setminus U)$ we may canonically identify $[[\phi]_U]_W$ and $[\phi]_{W \cup U}$, and thus $[\phi]_U$ is residually $(Y \setminus U)$ -nuclear.

Hence by assumption there exists a finite dimensional, continuous $(Y \setminus U)$ - C^* -algebra \mathfrak{D}_1 and $(Y \setminus U)$ -equivariant contractive c.p. maps

$$\psi_1: \mathfrak{A}/\mathfrak{A}(U) \rightarrow \mathfrak{D}_1, \quad \rho_1: \mathfrak{D}_1 \rightarrow \mathfrak{B}/\mathfrak{B}(U)$$

such that

$$\|[\phi]_U(p_{\mathfrak{A}}(a)) - \rho_1(\psi_1(p_{\mathfrak{A}}(a)))\| < \epsilon/6$$

for all $a \in F$, where $p_{\mathfrak{A}}: \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{A}(U)$ is the natural quotient map. Similarly, let $p_{\mathfrak{B}}: \mathfrak{B} \rightarrow \mathfrak{B}/\mathfrak{B}(U)$ be the natural quotient map.

Now we consider $\mathfrak{C}/\mathfrak{C}(U)$ as Y - C^* -algebras by $(\mathfrak{C}/\mathfrak{C}(U))(V) = \mathfrak{C}(V \cup U)/\mathfrak{C}(U)$ for $V \in \mathbb{O}(Y)$, and $\mathfrak{C} \in \{\mathfrak{A}, \mathfrak{B}\}$. Also, give \mathfrak{D}_1 a Y - C^* -algebra structure, by $\mathfrak{D}_1(V) := \mathfrak{D}_1(V \setminus U)$. This action is easily seen to be continuous, and clearly $\rho_1, \psi_1, p_{\mathfrak{A}}$ and $p_{\mathfrak{B}}$ are Y -equivariant.

Since \mathfrak{B} is continuous, we get

$$\mathfrak{B}(V \cup U)/\mathfrak{B}(U) = (\mathfrak{B}(V) + \mathfrak{B}(U))/\mathfrak{B}(U) \cong \mathfrak{B}(V)/(\mathfrak{B}(U) \cap \mathfrak{B}(V)) = \mathfrak{B}(V)/\mathfrak{B}(U \cap V).$$

Hence $0 \rightarrow \mathfrak{B}(U) \rightarrow \mathfrak{B} \xrightarrow{p_{\mathfrak{B}}} \mathfrak{B}/\mathfrak{B}(U) \rightarrow 0$ is a Y -equivariant extension, where we equip $\mathfrak{B}(U)$ with the Y - C^* -algebra structure $\mathfrak{B}(U)(V) = \mathfrak{B}(U \cap V)$. By Lemma 5.28 we may lift ρ_1 to a Y -equivariant contractive c.p. map $\tilde{\rho}_1: \mathfrak{D}_1 \rightarrow \mathfrak{B}$. Observe that

$$\|p_{\mathfrak{B}}(\phi(a) - \tilde{\rho}_1(\psi_1(p_{\mathfrak{A}}(a))))\| = \|[\phi]_U(p_{\mathfrak{A}}(a)) - \rho_1(\psi_1(p_{\mathfrak{A}}(a)))\| < \epsilon/6$$

for all $a \in F$. We may find a positive contractions $b \in \ker p_{\mathfrak{B}} = \mathfrak{B}(U)$, e.g. by picking b as an element of a quasi-central approximate identity, such that

$$\begin{aligned} \|(1-b)(\phi(a) - \tilde{\rho}_1(\psi_1(p_{\mathfrak{A}}(a))))\| &< \epsilon/6 \\ \|b\phi(a) - b^{1/2}\phi(a)b^{1/2}\| &< \epsilon/6 \\ \|(1-b)\tilde{\rho}_1(\psi_1(p_{\mathfrak{A}}(a))) - (1-b)^{1/2}\tilde{\rho}_1(\psi_1(p_{\mathfrak{A}}(a)))(1-b)^{1/2}\| &< \epsilon/6 \end{aligned}$$

for all $a \in F$. Let $\tilde{\rho} = (1-b)^{1/2}\tilde{\rho}_1(-)(1-b)^{1/2}$ which is also a Y -equivariant contractive c.p. lift of ρ_1 . By the above inequalities we have that

$$\|\phi(a) - (b^{1/2}\phi(a)b^{1/2} + \tilde{\rho}(\psi_1(p_{\mathfrak{A}}(a))))\| < \epsilon/2,$$

for all $a \in F$. Note that the contractive c.p. map $b^{1/2}\phi(-)b^{1/2}$ factors through $\mathfrak{B}(U)$. Let $V := Y \setminus \bar{U}$. Since $\mathfrak{B}(V) \cap \mathfrak{B}(U) = 0$ by continuity of \mathfrak{B} , it follows that $b^{1/2}\phi(-)b^{1/2}$ also has a factor $[\phi]_V: \mathfrak{A}/\mathfrak{A}(V) \rightarrow \mathfrak{B}/\mathfrak{B}(V)$ which is nuclear. Hence we may pick contractive c.p. maps

$$\psi_2: \mathfrak{A} \rightarrow \mathfrak{D}_2, \quad \rho_2: \mathfrak{D}_2 \rightarrow \mathfrak{B}(U),$$

with \mathfrak{D}_2 a finite dimensional C^* -algebra such that

$$\|b^{1/2}\phi(a)b^{1/2} - \rho_2(\psi_2(a))\| < \epsilon/2,$$

for all $a \in F$, such that $\rho_2(1_{\mathfrak{D}_2}) \leq b$ and such that ψ_2 factors through $\mathfrak{A}/\mathfrak{A}(\mathbf{V})$. Now, let $\mathfrak{D} := \mathfrak{D}_1 \oplus i_{\mathbf{U}}(\mathfrak{D}_2)$. Note that $i_{\mathbf{U}}(\mathfrak{D}_2)$ is a continuous \mathbf{Y} - C^* -algebra since \mathbf{U} is a non-empty, minimal open set. It follows that \mathfrak{D} is a continuous \mathbf{Y} - C^* -algebra, since it is the direct sum of two continuous \mathbf{Y} - C^* -algebras. Define

$$\psi := (\psi_1 \circ p_{\mathfrak{A}}, \psi_2): \mathfrak{A} \rightarrow \mathfrak{D}, \quad \rho = \langle \tilde{\rho}, \rho_2 \rangle: \mathfrak{D} \rightarrow \mathfrak{B}.$$

By the above estimates it clearly follows that

$$\|\phi(a) - \rho(\psi(a))\| < \epsilon$$

for all $a \in F$, and thus it remains to show that ψ and ρ are \mathbf{Y} -equivariant contractive c.p. maps. Clearly ψ is a contractive c.p. map since each $\psi_1 \circ p_{\mathfrak{A}}$ and ψ_2 is a contractive c.p. map. Moreover, ρ is clearly completely positive and since

$$\rho(1_{\mathfrak{D}}) = \tilde{\rho}(1) + \rho_2(1) = (1 - b)^{1/2} \tilde{\rho}_1(1)(1 - b)^{1/2} + \rho_2(1) \leq (1 - b) + b = 1,$$

it follows that ρ is contractive. It suffices to show that each $\psi_1 \circ p_{\mathfrak{A}}, \psi_2, \tilde{\rho}$ and ρ_2 is \mathbf{Y} -equivariant.

Clearly $\psi_1 \circ p_{\mathfrak{A}}$ is \mathbf{Y} -equivariant since each ψ_1 and $p_{\mathfrak{A}}$ is \mathbf{Y} -equivariant. Moreover, $\tilde{\rho}$ is \mathbf{Y} -equivariant as noted earlier in the proof. Since \mathfrak{B} is continuous (and thus a C^* -algebra over \mathbf{Y}) it follows from [MN09, Lemma 2.22] (which clearly also holds for c.p. maps) that ρ_2 is \mathbf{Y} -equivariant exactly if it factors through $\mathfrak{B}(\mathbf{U})$. This is satisfied by how we constructed the map. Since ψ_2 factors through $i_{\mathbf{U}}(\mathfrak{D}_2)$, it is \mathbf{Y} -equivariant if and only if $\psi_2(\mathfrak{A}(\mathbf{W})) = 0$ for all $\mathbf{W} \in \mathcal{O}(\mathbf{Y})$ for which $\mathbf{U} \not\subseteq \mathbf{W}$ if and only if $\psi_2(\mathfrak{A}(\mathbf{V})) = 0$ since \mathbf{U} is minimal. This is again satisfied by how we chose ψ_2 , which finishes the proof. \square

Corollary 5.30. *Let \mathbf{X} be a finite space, let \mathfrak{A} and \mathfrak{B} be \mathbf{X} - C^* -algebras, with \mathfrak{B} continuous, and let $\phi: \mathfrak{A} \rightarrow \mathfrak{B}$ be a c.p. map. Then ϕ is residually \mathbf{X} -nuclear if and only if the following holds: for any finite subset $F \subseteq \mathfrak{A}$, and any $\epsilon > 0$ there are \mathbf{X} -equivariant c.p. maps*

$$\psi: \mathfrak{A} \rightarrow \mathfrak{D}, \quad \rho: \mathfrak{D} \rightarrow \mathfrak{B},$$

such that

$$\|\phi(a) - \rho(\psi(a))\| < \epsilon, \quad \text{for all } a \in F,$$

where \mathfrak{D} is a finite dimensional, continuous \mathbf{X} - C^* -algebra, $\|\psi\| \leq 1$ and $\|\rho\| \leq \|\phi\|$.

Arveson's method of proving the Choi–Effros lifting theorem [Arv77], can also be used in the \mathbf{X} -equivariant case, if \mathbf{X} is finite.

Corollary 5.31. *Let \mathbf{X} be a finite space, \mathfrak{A} be a separable \mathbf{X} - C^* -algebra, $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{B} \xrightarrow{p} \mathfrak{B}/\mathfrak{J} \rightarrow 0$ be an extension of \mathbf{X} - C^* -algebras with $\mathfrak{B}/\mathfrak{J}$ continuous. Suppose that $\phi: \mathfrak{A} \rightarrow \mathfrak{B}/\mathfrak{J}$ is a contractive residually \mathbf{X} -nuclear map. Then there exists a contractive residually \mathbf{X} -nuclear map $\tilde{\phi}: \mathfrak{A} \rightarrow \mathfrak{B}$ such that $p \circ \tilde{\phi} = \phi$.*

Proof. Consider the cone

$$\mathcal{C}_1 := \{p \circ \psi \mid \psi \in CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B}), \|\psi\| \leq 1\},$$

i.e. the cone of all c.p. maps $\mathfrak{A} \rightarrow \mathfrak{B}/\mathfrak{J}$ which lift to a contractive residually \mathbf{X} -nuclear map $\mathfrak{A} \rightarrow \mathfrak{B}$. By Lemma 2.22 this is point-norm closed. So it suffices to show that ϕ

can be point-norm approximated by maps in \mathcal{C}_1 . By Corollary 5.30 we may approximate ϕ with X -equivariant c.p. maps which factor X -equivariantly through finite dimensional X - C^* -algebras by contractive maps. It follows from Lemma 5.28 that these maps are in \mathcal{C}_1 , and thus ϕ is in \mathcal{C}_1 . \square

The above Choi–Effros type lifting theorem implies the following.

Corollary 5.32. *The following hold.*

- (1) *Suppose that X is a finite space and that $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ is an extension of X - C^* -algebras. If \mathfrak{A} is separable and nuclear, then the extension has an X -equivariant contractive c.p. splitting.*
- (2) *Suppose that X is a finite space and that \mathfrak{A} is a separable, nuclear C^* -algebra over X . Then the functors $KK(X; \mathfrak{A}, -) \cong E(X; \mathfrak{A}, -)$ are naturally isomorphic.*

Proof. Ad (1): by applying the trick of Meyer–Nest (see proof of Lemma 5.28), we may assume that our X - C^* -algebras are continuous Y - C^* -algebras, for some larger (but still finite) space Y . Thus Corollary 5.31 implies the existence of a Y -equivariant contractive c.p. splitting, which is thus also an X -equivariant contractive c.p. splitting. Ad (2): this follows immediately from (1) above and [DM12, Corollary 5.3]. \square

6. PURELY LARGE EXTENSIONS OVER FINITE SPACES

In this section we prove that for sufficiently nice X - C^* -algebras \mathfrak{A} and \mathfrak{B} (e.g. when both are separable and continuous) the closed operator convex cone $CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})$ satisfies the (unital) purely large problem from Section 4, under the additional assumption that X is finite. In the case where every $\mathfrak{B}(U)$ has the corona factorisation property we prove that purely largeness may be replaced by the weaker condition of the extensions being X -full.

6.1. Absorbing $*$ -homomorphisms are X -full. We start by showing that if X is finite and \mathfrak{B} is suitably nice, then $CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})$ is generated by pure states in a certain sense. We need the following definition.

Definition 6.1. Let \mathfrak{B} be a σ -unital X - C^* -algebra. We say that \mathfrak{B} is X - σ -unital if $\mathfrak{B}(U)$ is σ -unital for every $U \in \mathbb{O}(X)$.

In particular, any separable X - C^* -algebra is X - σ -unital.

Recall, that if X is a finite T_0 space, and $x \in X$, then we let U^x denote the smallest open subset of X containing x .

Lemma 6.2. *Let X be finite T_0 space, and let \mathfrak{B} be a stable, X - σ -unital and continuous X - C^* -algebra. Fix strictly positive elements $h_x \in \mathfrak{B}(U^x)$ for $x \in X$. Then*

$$\mathcal{S} := \{\mathfrak{A} \xrightarrow{h_x \rho_x(-)} \mathfrak{B} \mid x \in X, \rho_x \text{ is a pure state on } \mathfrak{A}, \rho_x(\mathfrak{A}(X \setminus \overline{\{x\}})) = 0\}$$

generates $CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})$ as a closed operator convex cone.

In particular, if \mathfrak{A} is separable then $CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})$ is countably generated.

Proof. Let $h_x \rho_x(-) \in \mathcal{S}$, and let $a \in \mathfrak{A}(U)$. If $x \notin U$, then $U \subset X \setminus \overline{\{x\}}$ and thus $h_x \rho_x(a) = 0 \in \mathfrak{B}(U)$. If $x \in U$ then $U^x \subset U$, so $h_x \rho_x(a) \in \mathfrak{B}(U^x) \subset \mathfrak{B}(U)$. Thus $h_x \rho_x(-)$ is X -equivariant, and it is clearly residually X -nuclear, so $\mathcal{S} \subset CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})$.

By Corollary 5.30 any map in $CP_{\text{r-nuc}}(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ can be approximated by a composition $\mathfrak{A} \xrightarrow{\phi} \mathfrak{D} \xrightarrow{\psi} \mathfrak{B}$, where \mathfrak{D} is a finite dimensional, continuous X - C^* -algebra and ϕ and ψ are X -equivariant c.p. maps. By Lemma 5.27 we may decompose $\mathfrak{D} = \bigoplus_{x \in \mathsf{X}} i_x(\mathfrak{D}_x)$. Hence we may decompose ϕ and ψ into X -equivariant c.p. maps $\phi_x: \mathfrak{A} \rightarrow i_x(\mathfrak{D}_x)$ and $\psi_x: i_x(\mathfrak{D}_x) \rightarrow \mathfrak{B}$. It follows that $\psi \circ \phi = \sum_{x \in \mathsf{X}} \psi_x \circ \phi_x$. Hence it suffices to show that $\psi_x \circ \phi_x$ are in the closed operator convex cone generated by \mathcal{S} . Recall, as in [MN09, Lemma 2.22], that $CP(\mathsf{X}; \mathfrak{A}, i_x(\mathfrak{D}_x)) \cong CP(\mathfrak{A}/\mathfrak{A}(\mathsf{X} \setminus \overline{\{x\}}), \mathfrak{D}_x)$ and $CP(\mathsf{X}; i_x(\mathfrak{D}_x), \mathfrak{B}) \cong CP(\mathfrak{D}_x, \mathfrak{B}(\mathsf{U}^x))$ naturally. As in the classical case of nuclearity, we may assume that \mathfrak{D}_x is a matrix algebra, say M_{n_x} . Let $\psi'_x: \mathfrak{A}/\mathfrak{A}(\mathsf{X} \setminus \overline{\{x\}}) \rightarrow M_{n_x}$ and $\psi'_x: M_{n_x} \rightarrow \mathfrak{B}(\mathsf{U}^x)$ be the induced maps.

As in the first part of the proof of [EK01, Lemma 10], we may approximate $\psi'_x \circ \phi'_x$ by a sum of c.p. maps of the form

$$(6.1) \quad a' \mapsto \sum_{i,j=1}^{n_x} b_i'^* \rho'_x(a_i'^* a' a_j') b_j',$$

with $a'_1, \dots, a'_{n_x} \in \mathfrak{A}/\mathfrak{A}(\mathsf{X} \setminus \overline{\{x\}})$, $b'_1, \dots, b'_{n_x} \in \mathfrak{B}(\mathsf{U}^x)$ and ρ_x a pure state on $\mathfrak{A}/\mathfrak{A}(\mathsf{X} \setminus \overline{\{x\}})$. Since h_x is strictly positive in $\mathfrak{B}(\mathsf{U}^x)$ we may assume, by replacing b'_i with a small perturbation, that $b'_i = h_x^{1/2} b_i$ for some $b_i \in \mathfrak{B}(\mathsf{U}^x)$. Let $a_i \in \mathfrak{A}$ lifts of a'_i , and let ρ_x be a pure state on \mathfrak{A} which vanishes on $\mathfrak{A}(\mathsf{X} \setminus \overline{\{x\}})$ and lifts ρ'_x . Then the map in (6.1) corresponds to the X -equivariant c.p. map $\mathfrak{A} \rightarrow \mathfrak{B}$ given by

$$a \mapsto \sum_{i,j=1}^{n_x} b_i^* h_x \rho_x(a_i^* a a_j) b_j,$$

which is in the operator convex cone generated by \mathcal{S} . It follows that $\psi_x \circ \phi_x$ can be approximated by sums of maps on the above form, which in turn are in the closed operator convex cone generated by \mathcal{S} .

The “in particular” part follows since the pure state space of a separable C^* -algebra is separable in the weak- $*$ topology. Hence \mathcal{S} generates the same closed operator convex cone as a set where we only choose countably many weak- $*$ dense pure states. \square

The following proposition shows that in nice cases when $\mathcal{C} = CP_{\text{r-nuc}}(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$, we may describe the map $\mathfrak{B}_{\mathcal{C}}$ in terms of the actions of X on \mathfrak{A} and \mathfrak{B} . This gives a solution to Question 4.15 for a large class of closed operator convex cones.

Proposition 6.3. *Let X be a finite space and let \mathfrak{A} and \mathfrak{B} be X - C^* -algebras with \mathfrak{A} separable and lower semicontinuous, and \mathfrak{B} stable, X - σ -unital, and continuous, and let $\mathcal{C} = CP_{\text{r-nuc}}(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$. Then $\mathfrak{B}_{\mathcal{C}}(a) = \mathfrak{B}(\mathsf{U}_a)$ for all $a \in \mathfrak{A}$.*

Moreover, $\mathfrak{B}_{\mathcal{C}} = \mathfrak{B}_{CP(\mathsf{X}; \mathfrak{A}, \mathfrak{B})}$.

Proof. Let $a \in \mathfrak{A}$. By Lemma 6.2, \mathcal{C} is countably generated. Thus it follows from Lemma 4.7 that $\mathfrak{B}_{\mathcal{C}}(a) = \mathfrak{B}_{\mathcal{C}}(a^*a)$. Since a and a^*a are contained in exactly the same two-sided, closed ideals in \mathfrak{A} , it follows that $\mathsf{U}_a = \mathsf{U}_{a^*a}$, so we may assume, without loss of generality, that $a \in \mathfrak{A}$ is positive.

We may assume that X is a T_0 space by Remark 5.3. For $x \in \mathsf{X}$ let U^x denote the smallest open set which contains x , and let V^x denote the largest open set which does not contain x . For each x fix a strictly positive element b_x in $\mathfrak{B}(\mathsf{U}^x)$, and a state $\phi_x: \mathfrak{A} \rightarrow \mathbb{C}$ such

that $\phi_x(\mathfrak{A}(\mathbf{V}^x)) = 0$ and such that the induced state on $\mathfrak{A}/\mathfrak{A}(\mathbf{V}^x)$ is faithful if $\mathfrak{A}/\mathfrak{A}(\mathbf{V}^x)$ is non-zero. Now define a map

$$\phi: \mathfrak{A} \rightarrow \mathfrak{B}, \quad \phi(c) = \sum_{x \in \mathbf{X}} \phi_x(c) b_x.$$

As $\phi_x(-) b_x = b_x^{1/2} \phi_x(-) b_x^{1/2}$ is completely positive for each x , ϕ is completely positive. For any $x \notin \mathbf{U}_c$ we have $\mathbf{U}_c \subset \mathbf{V}^x$. Hence $c \in \mathfrak{A}(\mathbf{V}^x)$ which implies $\phi_x(c) = 0$. It follows that

$$\phi(c) = \sum_{x \in \mathbf{U}_c} \phi_x(c) b_x \in \sum_{x \in \mathbf{U}_c} \mathfrak{B}(\mathbf{U}^x) = \mathfrak{B}(\bigcup_{x \in \mathbf{U}_c} \mathbf{U}^x) = \mathfrak{B}(\mathbf{U}_c).$$

Hence ϕ is \mathbf{X} -equivariant. We want to show that $\phi(a)$ is full in $\mathfrak{B}(\mathbf{U}_a)$.

If $\mathbf{U}_a = \emptyset$ then $\mathfrak{B}(\mathbf{U}_a) = 0$, so $\phi(a) = 0$ is full in $\mathfrak{B}(\mathbf{U}_a)$. Thus we may consider the case where \mathbf{U}_a is non-empty. Suppose for contradiction that there is $x \in \mathbf{U}_a$ such that $\mathfrak{A}(\mathbf{V}^x) = \mathfrak{A}$. Then $a \in \mathfrak{A}(\mathbf{U}_a) \cap \mathfrak{A}(\mathbf{V}^x) = \mathfrak{A}(\mathbf{U}_a \cap \mathbf{V}^x)$. However, since $\mathbf{U}_a \cap \mathbf{V}^x$ does not contain x , it is a proper subset of \mathbf{U}_a , which contradicts that a is \mathbf{U}_a -full. Hence $\mathfrak{A}/\mathfrak{A}(\mathbf{V}^x)$ is non-zero for all $x \in \mathbf{U}_a$. By construction $\phi_x(a) > 0$ for all $x \in \mathbf{U}_a$ and thus $\phi(a)$ is full in $\mathfrak{B}(\mathbf{U}_a)$. Moreover, ϕ is obviously residually \mathbf{X} -nuclear. Hence $\mathfrak{B}(\mathbf{U}_a) = \mathfrak{B}_c(a)$ by Lemma 5.15.

That $\mathfrak{B}_c = \mathfrak{B}_{CP(\mathbf{X}; \mathfrak{A}, \mathfrak{B})}$ follows, since

$$\mathfrak{B}_c(a) \subset \mathfrak{B}_{CP(\mathbf{X}; \mathfrak{A}, \mathfrak{B})}(a) \subset \mathfrak{B}(\mathbf{U}_a) = \mathfrak{B}_c(a)$$

for all $a \in \mathfrak{A}$. □

The rest of this subsection is dedicated to showing, that \mathbf{X} -equivariant $CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ -absorbing $*$ -homomorphisms are \mathbf{X} -full.

Note that if \mathfrak{J} is a two-sided, closed ideal in \mathfrak{B} , then \mathfrak{J} is an essential ideal in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$. Hence there is an induced injective $*$ -homomorphism $\iota: \mathcal{M}(\mathfrak{B}, \mathfrak{J}) \hookrightarrow \mathcal{M}(\mathfrak{J})$.

Lemma 6.4. *The image $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J})) \subset \mathcal{M}(\mathfrak{J})$ is a hereditary C^* -subalgebra.*

Proof. If $m \in \mathcal{M}(\mathfrak{J})$ and $m_1, m_2 \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ let $x \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ be defined by

$$xb = m_1(m(m_2b)), \quad bx = ((bm_1)m)m_2$$

for $b \in \mathfrak{B}$. It is easily seen that x is well-defined and that $\iota(x) = \iota(m_1)m\iota(m_2)$. Hence $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J})) \subset \mathcal{M}(\mathfrak{J})$ is a hereditary C^* -subalgebra. □

Lemma 6.5. *Let \mathfrak{B} be a σ -unital, stable C^* -algebra, let $m \in \mathcal{M}(\mathfrak{B})$, and let m_∞ be an infinite repeat of m . Then m_∞ is full in $\mathcal{M}(\mathfrak{B}, \overline{\mathfrak{B}m\mathfrak{B}})$ and $\iota(m_\infty)$ is full in $\mathcal{M}(\overline{\mathfrak{B}m\mathfrak{B}})$.*

In particular, if \mathfrak{J} is a two-sided, closed ideal in \mathfrak{B} , and there exists an $m \in \mathcal{M}(\mathfrak{B})$ with $\mathfrak{J} = \overline{\mathfrak{B}m\mathfrak{B}}$, then $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J}))$ is a full, hereditary C^ -subalgebra of $\mathcal{M}(\mathfrak{J})$.*

Proof. By replacing m with m^*m we may assume that m is positive. The infinite repeat m_∞ is clearly an element in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$. By [Kuc04, Lemma 14] it follows that $\iota(m_\infty)$ is full in $\mathcal{M}(\mathfrak{J})$, since it is also an infinite repeat in $\mathcal{M}(\mathfrak{J})$. Since $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J}))$ is a hereditary C^* -subalgebra of $\mathcal{M}(\mathfrak{J})$ by Lemma 6.4 it follows that m_∞ is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$.

The “in particular” part follows from Lemma 6.4 and since $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J}))$ contains an element which is full in $\mathcal{M}(\mathfrak{J})$. □

Very similar to Lemma 4.7 we have the following lemma.

Lemma 6.6. *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras, with \mathfrak{A} separable, \mathfrak{B} σ -unital and stable, and let $\mathcal{C} \subset CP(\mathfrak{A}, \mathfrak{B})$ be a countably generated, closed operator convex cone. If $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is a \mathcal{C} -absorbing $*$ -homomorphism weakly in \mathcal{C} , then*

$$\mathcal{M}(\mathfrak{B}, \mathfrak{B}_{\mathcal{C}}(a)) = \overline{\mathcal{M}(\mathfrak{B})\Phi(a)\mathcal{M}(\mathfrak{B})}$$

for every $a \in \mathfrak{A}$.

Moreover, if \mathfrak{A} is unital, \mathcal{C} is non-degenerate and $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is a unital \mathcal{C} -absorbing $*$ -homomorphism weakly in \mathcal{C} , then

$$\mathcal{M}(\mathfrak{B}, \mathfrak{B}_{\mathcal{C}}(a)) = \overline{\mathcal{M}(\mathfrak{B})\Phi(a)\mathcal{M}(\mathfrak{B})}$$

for every $a \in \mathfrak{A}$.

Proof. The unital and non-unital case are identical, so we only prove the non-unital case. Since $\Phi(a)$ and $\Phi(a^*a)$ generate the same two-sided, closed ideal, and since $\mathfrak{B}_{\mathcal{C}}(a) = \mathfrak{B}_{\mathcal{C}}(a^*a)$ by Lemma 4.7, we may restrict to the case where $a \in \mathfrak{A}$ is positive.

Let t_1, t_2, \dots be isometries in $\mathcal{M}(\mathfrak{B})$ such that $\sum_{n=1}^{\infty} t_n t_n^*$ converges strictly to 1. The infinite repeat $\Phi_{\infty} = \sum_{n=1}^{\infty} t_n \Phi(-) t_n^*$ is also \mathcal{C} -absorbing and weakly in \mathcal{C} , and

$$\overline{\mathcal{M}(\mathfrak{B})\Phi(a)\mathcal{M}(\mathfrak{B})} = \overline{\mathcal{M}(\mathfrak{B})\Phi_{\infty}(a)\mathcal{M}(\mathfrak{B})},$$

for all $a \in \mathfrak{A}$, since Φ and Φ_{∞} are asymptotically unitarily equivalent. Note that $\mathfrak{B}_{\mathcal{C}}(a) = \overline{\mathfrak{B}\Phi_{\infty}(a)\mathfrak{B}}$ by Lemma 4.7. By Lemma 6.5, $\Phi_{\infty}(a)$ is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{B}_{\mathcal{C}}(a))$. \square

Corollary 6.7. *Let X be a finite space and let \mathfrak{A} and \mathfrak{B} be X - C^* -algebras with \mathfrak{A} separable and lower semicontinuous, and \mathfrak{B} stable, X - σ -unital, and continuous, and let $\mathcal{C} = CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})$. Then there exists a \mathcal{C} -absorbing $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ weakly in \mathcal{C} , and any such $*$ -homomorphism is X -full.*

Moreover, if \mathfrak{A} is unital and $\mathfrak{B}(U_{1_{\mathfrak{A}}}) = \mathfrak{B}$, then there exists a unital \mathcal{C} -absorbing $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ weakly in \mathcal{C} , and any such $*$ -homomorphism is X -full.

Proof. For the unital case, note that $\mathfrak{B}_{\mathcal{C}}(1_{\mathfrak{A}}) = \mathfrak{B}(U_{1_{\mathfrak{A}}}) = \mathfrak{B}$ by Proposition 6.3. This is equivalent to \mathcal{C} being non-degenerate.

Thus the existence of a (unital) \mathcal{C} -absorbing $*$ -homomorphism Φ weakly in \mathcal{C} follows from Lemma 6.2 and Theorem 3.14. By Proposition 6.3 and Lemma 6.6 it follows that

$$\overline{\mathcal{M}(\mathfrak{B})\Phi(a)\mathcal{M}(\mathfrak{B})} = \mathcal{M}(\mathfrak{B}, \mathfrak{B}_{\mathcal{C}}(a)) = \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U_a)) = \mathcal{M}(\mathfrak{B})(U_a)$$

for every $a \in \mathfrak{A}$. Thus Φ is X -full. \square

6.2. The purely large problem over finite spaces. In [EK01] an important part of the proof of the main theorem was, that every purely large extension had a certain purely infinite type comparison property. The following lemma seems to be the closest we can get to a similar result with respect to closed operator convex cones.

Lemma 6.8. *Let $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ be a \mathcal{C} -purely large extension. Let $\epsilon > 0$, $x \in \mathfrak{E}$ be positive with $\|x\| = 1$, and let that $b \in \mathfrak{B}$ with $\|b\| = 1$. Suppose that $b \in \mathfrak{B}_{\mathcal{C}}(p(g(x)))$ for any positive, continuous function $g: [0, 1] \rightarrow [0, 1]$ with $g(0) = 0$ and $g(1) = 1$. Then there is a $b_0 \in \mathfrak{B}$ with $\|b_0\| \leq 1$ such that*

$$\|b - b_0^* x b_0\| < \epsilon.$$

Proof. Let $f, g: [0, 1] \rightarrow [0, 1]$ be the continuous functions

$$f(t) = \begin{cases} 0, & \text{for } t = 0 \\ \text{affine}, & \text{for } 0 \leq t \leq 1 - \epsilon/3 \\ 1, & \text{for } 1 - \epsilon/3 \leq t \leq 1 \end{cases}, \quad g(t) = \begin{cases} 0, & \text{for } 0 \leq t \leq 1 - \epsilon/3 \\ \text{affine}, & \text{for } 1 - \epsilon/3 \leq t \leq 1 \\ 1, & \text{for } t = 1. \end{cases}$$

Then $\|f(x) - x\| \leq \epsilon/3$ and $f(x)g(x) = g(x)f(x) = g(x)$. Since the extension is purely large with respect to \mathfrak{B}_c , the C^* -algebra $g(x)\mathfrak{B}g(x)$ contains a stable, σ -unital C^* -subalgebra \mathfrak{D} which is full in $\mathfrak{B}_c(p(g(x)))$. Since $b \in \mathfrak{B}_c(p(g(x)))$ we may, as in the proof of [EK01, Lemma 7], find $d \in \mathfrak{D}_+$, $b_1, \dots, b_n \in \mathfrak{B}$ such that $\|b - \sum b_i^* d b_i\| < \epsilon/3$, and $V_1, \dots, V_n, P \in \mathcal{M}(\mathfrak{D})$ such that P is a projection for which $Pd = d$, and $V_i^* V_j = \delta_{ij} P$. Define

$$b'_0 := \sum_{i=1}^n V_i d^{1/2} b_i, \quad b_0 = b'_0 / \|b'_0\|.$$

We get that

$$b_0^* b'_0 = \sum_{i,j=1}^n b_i^* d^{1/2} V_i^* V_j d^{1/2} b_j = \sum_{i=1}^n b_i^* d b_i.$$

Thus $\|b_0^* b'_0 - b\| < \epsilon/3$ which implies, since $\|b\| = 1$, that $|\|b_0^* b'_0\| - 1| < \epsilon/3$. Hence

$$\|b_0^* b_0 - b_0^* b'_0\| = \left\| b_0^* b'_0 \left(\frac{1}{\|b_0^* b'_0\|} - 1 \right) \right\| \leq |1 - \|b_0^* b'_0\|| < \epsilon/3.$$

Note that $f(x)d' = d'f(x) = d'$ for any $d' \in \mathfrak{D}$. Hence $b_0^* f(x)b_0 = b_0^* b_0$ and thus

$$b_0^* x b_0 \approx_{\epsilon/3} b_0^* f(x)b_0 = b_0^* b_0 \approx_{\epsilon/3} b_0^* b'_0 \approx_{\epsilon/3} b.$$

□

Given an X - C^* -algebra \mathfrak{A} , we may give the forced unitisation \mathfrak{A}^\dagger an X - C^* -algebra structure by letting $\mathfrak{A}^\dagger(\mathbf{U}) = \mathfrak{A}(\mathbf{U})$ for $\mathbf{U} \in \mathcal{O}(X)$, when $\mathbf{U} \neq X$, and $\mathfrak{A}^\dagger(X) = \mathfrak{A}^\dagger$. Note that this construction may ruin certain properties which the action of X on \mathfrak{A} had, e.g. (finite) upper semicontinuity. However, lower semicontinuity will be preserved.

We need the following lemma. For notation see Lemma 3.9.

Lemma 6.9. *Let X be a topological space, and let \mathfrak{A} and \mathfrak{B} be X - C^* -algebras. Suppose that $\mathfrak{B}(X) = \mathfrak{B}$. Then*

$$CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})^\dagger = CP_{\text{r-nuc}}(X; \mathfrak{A}^\dagger, \mathfrak{B}).$$

Proof. If $\phi \in CP_{\text{r-nuc}}(X; \mathfrak{A}^\dagger, \mathfrak{B})$ then $\phi|_{\mathfrak{A}}$ is residually X -nuclear. In fact, if $\mathbf{U} \in \mathcal{O}(X) \setminus \{X\}$, then the induced map $\mathfrak{A}^\dagger/\mathfrak{A}^\dagger(\mathbf{U}) \rightarrow \mathfrak{B}/\mathfrak{B}(\mathbf{U})$ is nuclear. Since $\mathfrak{A}^\dagger/\mathfrak{A}^\dagger(\mathbf{U}) \cong (\mathfrak{A}/\mathfrak{A}(\mathbf{U}))^\dagger$, the restriction to $\mathfrak{A}/\mathfrak{A}(\mathbf{U})$ is nuclear. Hence $\phi \in CP_{\text{r-nuc}}(X; \mathfrak{A}, \mathfrak{B})^\dagger$.

Now let us show, that a positive map $\phi: \mathfrak{A}^\dagger \rightarrow \mathfrak{B}$ is nuclear if $\phi|_{\mathfrak{A}}$ is nuclear. Given a finite subset $F \subset \mathfrak{A}^\dagger$ and $\epsilon > 0$, we may assume that $F = F' \cup \{1\}$ where $F' \subset \mathfrak{A}$. Let $a \in \mathfrak{A}$ be a positive contraction such that $\|aba - b\| < \epsilon/(2\|\phi\|)$ for all $b \in F'$. Let $\tilde{\phi}: \mathfrak{A} \rightarrow \mathfrak{B}$ be a c.p. map factoring by c.p. maps through a matrix algebra, such that $\|\tilde{\phi}\| \leq \|\phi\|$ and $\|\tilde{\phi}(aba) - \phi(aba)\| < \epsilon/2$ for all $b \in F$. Now let ψ be the composition $\mathfrak{A}^\dagger \twoheadrightarrow \mathbb{C} \rightarrow \mathfrak{B}$ given by $\psi(1) = \phi(1 - a^2)$, which is completely positive since ϕ is positive. Then the sum $(\tilde{\phi} \circ \text{Ad } a) + \psi$ factors through a finite dimensional C^* -algebra and approximates ϕ up to ϵ on F . Hence ϕ is nuclear.

Let $\phi \in CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B})^\dagger$ and $\mathbf{U} \in \mathbb{O}(\mathbf{X}) \setminus \{\mathbf{X}\}$. Clearly ϕ is \mathbf{X} -equivariant. Since $\phi|_{\mathfrak{A}}$ is residually \mathbf{X} -nuclear, it follows that the restriction of $[\phi]_{\mathbf{U}}: \mathfrak{A}^\dagger/\mathfrak{A}^\dagger(\mathbf{U}) \rightarrow \mathfrak{B}/\mathfrak{B}(\mathbf{U})$ to $\mathfrak{A}/\mathfrak{A}(\mathbf{U})$ is nuclear. Since $\mathfrak{A}^\dagger/\mathfrak{A}^\dagger(\mathbf{U}) \cong (\mathfrak{A}/\mathfrak{A}(\mathbf{U}))^\dagger$, it follows by what we proved above, that $[\phi]_{\mathbf{U}}$ is nuclear. Hence ϕ is residually \mathbf{X} -nuclear. \square

Definition 6.10. We will say that an extension of \mathbf{X} - C^* -algebras \mathfrak{A} by \mathfrak{B} is *\mathbf{X} -purely large* if it is $CP(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ -purely large.

Moreover, we say that an extension of \mathfrak{A} by \mathfrak{B} of \mathbf{X} - C^* -algebras is *weakly residually \mathbf{X} -nuclear* if it is a $CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ -extension.

It follows immediately from Proposition 6.3 that if \mathbf{X} is finite, \mathfrak{A} is a separable, lower semicontinuous \mathbf{X} - C^* -algebra, and \mathfrak{B} is a stable, \mathbf{X} - σ -unital, continuous \mathbf{X} - C^* -algebra, then an extension of \mathfrak{A} by \mathfrak{B} is \mathbf{X} -purely large if and only if it is $CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ -purely large.

The following is one of our main theorems. It gives a solution to the purely large problem (Question 4.11) for a large class of closed operator convex cones, showing that they *do* satisfy the (unital) purely large problem. Note that we do *not* assume any infinity nor nuclearity criteria on our C^* -algebras (as done in Proposition 4.18).

Theorem 6.11. *Let \mathbf{X} be a finite space, and let $\epsilon: 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{C} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras. Suppose that \mathfrak{A} is a separable, lower semicontinuous \mathbf{X} - C^* -algebra, and \mathfrak{B} is a stable, \mathbf{X} - σ -unital, continuous \mathbf{X} - C^* -algebra.*

Then ϵ absorbs any trivial, weakly residually \mathbf{X} -nuclear extension if and only if ϵ is \mathbf{X} -purely large and absorbs the zero extension.

If, in addition, ϵ is unital and $\mathfrak{B}(\mathbf{U}_{1_{\mathfrak{A}}}) = \mathfrak{B}$, then ϵ absorbs any trivial, unital weakly residually \mathbf{X} -nuclear extension if and only if ϵ is \mathbf{X} -purely large.

Proof. Let $\mathcal{C} := CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$. The “only if” part follows from Proposition 4.10, where we note that, in the unital case, $\mathfrak{B}_{\mathcal{C}}(1_{\mathfrak{A}}) = \mathfrak{B}(\mathbf{U}_{1_{\mathfrak{A}}}) = \mathfrak{B}$ by Proposition 6.3, which is equivalent to \mathcal{C} being non-degenerate.

To prove the “if” part, it follows from Proposition 4.17, Lemma 6.9 and the fact that \mathfrak{A}^\dagger is also lower semicontinuous, that it suffices to prove the unital version of the theorem. Thus assume that ϵ is unital and $\mathfrak{B}(\mathbf{U}_{1_{\mathfrak{A}}}) = \mathfrak{B}$ and that the extension is \mathbf{X} -purely large.

Let $\sigma: \mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ and $\tau: \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$ be the canonical $*$ -homomorphisms induced by the extension ϵ . Let $\mathfrak{C} \subset \mathfrak{E}$ be a separable C^* -subalgebra with $1_{\mathfrak{E}} \in \mathfrak{C}$, containing a strictly positive element of \mathfrak{B} , such that $\mathfrak{C} + \mathfrak{B} = \mathfrak{E}$, and let $p_0 = p|_{\mathfrak{C}}: \mathfrak{C} \rightarrow \mathfrak{A}$, which is surjective.

We may assume that \mathbf{X} is a T_0 space by Remark 5.3. Let \mathcal{S} be as in Lemma 6.2. By Lemma 2.24

$$\mathcal{C}_0 = \{\phi \circ p_0 : \phi \in \mathcal{C}\} \subset CP(\mathfrak{C}, \mathfrak{B})$$

is a closed operator convex cone generated by the set $\mathcal{S}_0 := \{\phi' \circ p_0 : \phi' \in \mathcal{S}\}$. We will show that the unital $*$ -homomorphism $\sigma_0 := \sigma|_{\mathfrak{C}}: \mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ approximately dominates any map $\phi \in \mathcal{S}_0$.

Let $\phi = \phi' \circ p_0$ with $\phi' \in \mathcal{S}$. Write $\phi' = h_x \rho'_x(-)$ for some $x \in \mathbf{X}$ with h_x a strictly positive contraction in $\mathfrak{B}(\mathbf{U}^x)$, and ρ'_x a pure state on \mathfrak{A} vanishing on $\mathfrak{A}(\mathbf{X} \setminus \overline{\{x\}})$. Note that $\rho_x = \rho'_x \circ p_0$ is a pure state on \mathfrak{C} since p_0 is a surjective $*$ -homomorphism. Fix $F \subset \mathfrak{C}$ a finite set of contractions and $\epsilon > 0$. By [AAP86], we may excise ρ_x as follows. There is a positive $d \in \mathfrak{C}$ such that $\rho_x(d) = \|d\| = 1$, and

$$\|d^2 \rho_x(c) - dcd\| < \epsilon/2,$$

for all $c \in F$. Since $\rho_x(d) = \|d\| = 1$ it follows (e.g. by considering the GNS representation induced by ρ_x), that for any continuous function $g: [0, 1] \rightarrow [0, 1]$ for which $g(0) = 0$ and $g(1) = 1$, we have $\rho_x(g(d)) > 0$. Since \mathfrak{A} is a lower semicontinuous \mathbf{X} - C^* -algebra, it follows that $p_0(g(d^2))$ is \mathbf{U}_g -full, for some unique open subset \mathbf{U}_g of \mathbf{X} . Note that $\mathfrak{B}_c(p_0(g(d))) = \mathfrak{B}(\mathbf{U}_g)$ by Proposition 6.3. If we suppose that $\mathbf{U}^x \not\subset \mathbf{U}_g$ for some g , i.e. $x \notin \mathbf{U}_g$, then $\mathbf{U}_g \subset \mathbf{X} \setminus \overline{\{x\}}$ which would imply that $p_0(g(d^2)) \in \mathfrak{A}(\mathbf{X} \setminus \overline{\{x\}})$ and thus $\rho_x(g(d^2)) = 0$. However, this is false, so we must have that $\mathbf{U}^x \subset \mathbf{U}_g$ for all g . Since $h_x \in \mathfrak{B}(\mathbf{U}^x) \subset \mathfrak{B}(\mathbf{U}_g)$ for any g as above, we may apply Lemma 6.8 to obtain a contraction $b_0 \in \mathfrak{B}$ such that $h_x \approx_{\epsilon/2} b_0^* d^2 b_0$. Defining $b := db_0$ we get that $\|b\| \leq 1$ and

$$h_x \rho_x(c) \approx_{\epsilon/2} b_0^* d \rho_x(c) db_0 \approx_{\epsilon/2} b_0^* d c d b_0 = b^* c b = b^* \sigma_0(c) b$$

for all $c \in F$. Hence σ_0 approximately dominates $\phi = h_x \rho_x(-)$.

Let (b_n) be a bounded sequence in \mathfrak{B} such that $\|b_n^* \sigma_0(c) b_n - \phi(c)\| \rightarrow 0$ for every $c \in \mathfrak{C}$. Since \mathfrak{C} contains a strictly positive element h for \mathfrak{B} , and since $\phi(h) = (h_x \rho'_x(-)) \circ p_0(h) = 0$, we get that $\|b_n^* \sigma_0(h) b_n\| = \|b_n^* h b_n\| \rightarrow 0$. It follows that $\|b_n^* b b_n\| \rightarrow 0$ for every $b \in \mathfrak{B}$, and thus σ_0 *strongly* approximately dominates ϕ .

Since σ_0 strongly approximately dominates any c.p. map in \mathcal{S}_0 , it follows from Theorems 3.12 and 3.4 that $\sigma_0 \oplus \Psi \sim_{ap} \sigma_0$ for any unital $*$ -homomorphism $\Psi: \mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ weakly in \mathcal{C}_0 .

Let $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ be a unital \mathcal{C} -absorbing $*$ -homomorphism weakly in \mathcal{C} of Theorem 3.14 which exists by Lemma 6.2. Then $\Phi \circ p_0$ is unital and weakly in \mathcal{C}_0 and thus $\sigma_0 \oplus (\Phi \circ p_0) \sim_{ap} \sigma_0$. In particular, there is a unitary $u \in \mathcal{M}(\mathfrak{B})$ such that

$$\tau(p_0(c)) = \pi(\sigma_0(c)) = \pi(u^*(\sigma_0(c) \oplus \Phi(p_0(c)))u) = \pi(u)^*(\tau(p_0(c)) \oplus (\pi \circ \Phi)(p_0(c)))\pi(u)$$

for all $c \in \mathfrak{C}$. Since p_0 is surjective, we get that \mathfrak{e} absorbs the trivial, unital \mathcal{C} -extension with Busby map $\pi \circ \Phi$. Since this latter extension absorbs all trivial, unital \mathcal{C} -extensions, it follows that \mathfrak{e} absorbs all trivial, unital \mathcal{C} -extensions. \square

Remark 6.12. If \mathfrak{A} is exact (or even locally reflexive), then the conditions that \mathfrak{A} is lower semicontinuous and \mathfrak{B} is continuous, are redundant.

In fact, as in [MN09, Section 2.9] we may find a larger, but still finite, space \mathbf{Y} acting on \mathfrak{A} and \mathfrak{B} continuously, such that $CP(\mathbf{X}; \mathfrak{A}, \mathfrak{B}) = CP(\mathbf{Y}; \mathfrak{A}, \mathfrak{B})$. Thus an extension is \mathbf{X} -purely large if and only if it is \mathbf{Y} -purely large. By Remark 5.25 it follows that the residually \mathbf{X} -nuclear maps are exactly the residually \mathbf{Y} -nuclear maps. Since finite sums and finite intersections of σ -unital, two-sided, closed ideals are again σ -unital, it follows that if \mathfrak{B} is \mathbf{X} - σ -unital then it is also \mathbf{Y} - σ -unital. Thus we may replace \mathbf{X} with \mathbf{Y} and the result follows.

For applications of the above theorem in the non-unital case, it can be hard to determine whether or not an extension absorbs the zero extension. However, when the quotient algebra is sufficiently non-unital with a sufficiently nice action of \mathbf{X} , we get this for free by knowing that the extension is \mathbf{X} -purely large. Note that we assume in the following corollary that the extension is \mathbf{X} -equivariant, which was *not* a part of the assumptions in Theorem 6.11.

Corollary 6.13. *Let \mathbf{X} be a finite space, let $\mathfrak{e}: 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{C} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of \mathbf{X} - C^* -algebras such that \mathfrak{A} is separable and lower semicontinuous, and \mathfrak{B} is stable, \mathbf{X} - σ -unital and continuous. Suppose that $\mathfrak{A}/\mathfrak{A}(\mathbf{U})$ is non-zero and non-unital for all $\mathbf{U} \in \mathcal{O}(\mathbf{X}) \setminus \{\mathbf{X}\}$.*

Then \mathfrak{e} absorbs any trivial, weakly residually \mathbf{X} -nuclear extension if and only if \mathfrak{e} is \mathbf{X} -purely large.

Proof. Let $\mathcal{C} := CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$. One part follows from Proposition 4.10 so suppose that \mathfrak{e} is \mathbf{X} -purely large. By Theorem 6.11, Proposition 4.17 and Lemma 6.9, \mathfrak{e} absorbs any trivial \mathcal{C} -extension if and only if the unitised extension $\mathfrak{e}^\dagger : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E}^\dagger \xrightarrow{p^\dagger} \mathfrak{A}^\dagger \rightarrow 0$ is \mathbf{X} -purely large. So we will show that this is the case.

Let $y \in \mathfrak{E}^\dagger$. Note that if $a \in \mathfrak{A} \subset \mathfrak{A}^\dagger$, then a is U_a -full for the same U_a regardless of whether we consider a as an element of the \mathbf{X} - C^* -algebra \mathfrak{A} or of the \mathbf{X} - C^* -algebra \mathfrak{A}^\dagger . Thus if $y \in \mathfrak{E}$, then

$$\mathfrak{B}_{\mathfrak{e}}(p(y)) = \mathfrak{B}_{\mathfrak{e}^\dagger}(p^\dagger(y)) = \mathfrak{B}(U_{p^\dagger(y)}) = \mathfrak{B}(U_{p(y)})$$

by Lemma 4.16 and Proposition 6.3. It follows that $\overline{y\mathfrak{B}y^*}$ contains a stable σ -unital C^* -subalgebra which is full in $\mathfrak{B}(U_{p^\dagger(y)})$, since \mathfrak{e} is \mathbf{X} -purely large.

Hence it remains to consider the case $y \in \mathfrak{E}^\dagger \setminus \mathfrak{E}$. Without loss of generality we may assume that $y = 1 - x$ for an $x \in \mathfrak{E}$. Note that $U_{p^\dagger(1-x)} = \mathbf{X}$. Suppose that $(1-x)\mathfrak{E} \subset \mathfrak{B} + \mathfrak{E}(U)$ for some $U \neq \mathbf{X}$. Then $x + (\mathfrak{B} + \mathfrak{E}(U))$ would be a unit for $\mathfrak{E}/(\mathfrak{B} + \mathfrak{E}(U)) \cong \mathfrak{A}/\mathfrak{A}(U)$, which is a contradiction to the assumption that $\mathfrak{A}/\mathfrak{A}(U)$ is non-zero and non-unital for $U \in \mathbb{O}(\mathbf{X}) \setminus \{\mathbf{X}\}$. Hence we may find positive elements x'_U in \mathfrak{E} such that $(1-x)x'_U \notin \mathfrak{B} + \mathfrak{E}(U)$. Let $x' = \sum x'_U$. It is easily seen that $z := (1-x)x' \notin \mathfrak{B} + \mathfrak{E}(U)$ for any $U \neq \mathbf{X}$, and thus $p(z) \notin \mathfrak{A}(U)$. Hence $U_{p(z)} = U_{p^\dagger(1-x)} = \mathbf{X}$, and thus $\overline{z\mathfrak{B}z^*}$ contains a stable σ -unital C^* -subalgebra which is full in \mathfrak{B} since \mathfrak{e} is \mathbf{X} -purely large. Since $\overline{z\mathfrak{B}z^*} \subset \overline{(1-x)\mathfrak{B}(1-x)^*}$ it follows that \mathfrak{e}^\dagger is \mathbf{X} -purely large. \square

Remark 6.14. One might find it interesting to compare the assumptions on \mathfrak{A} above to the similar assumptions made by Kirchberg in [Kir00, Hauptsatz 4.2]. Here Kirchberg always assumes that \mathfrak{A} is separable, stable (and thus has no unital quotients), that $\mathfrak{A}(U) = \mathfrak{A}$ only when $U = \mathbf{X}$ (corresponding to $\mathfrak{A}/\mathfrak{A}(U)$ is non-zero for $U \neq \mathbf{X}$) and that the action of \mathbf{X} on \mathfrak{A} is monotone continuous (which is the same as lower semicontinuous when \mathbf{X} is finite). Kirchberg also assumes that \mathfrak{A} is exact, which is not needed above.

6.3. The corona factorisation property. Checking that an extension is \mathbf{X} -purely large can be very hard in general. This was the motivation, in the classical case, to introduce the *corona factorisation property* (see [Kuc06] and [KN06b]). Recall that a stable C^* -algebra \mathfrak{B} has the corona factorisation property if any norm-full projection $P \in \mathcal{M}(\mathfrak{B})$ is properly infinite, or equivalently, Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathfrak{B})}$.

Under the additional assumption on \mathfrak{B} that $\mathfrak{B}(U)$ has the corona factorisation property for each $U \in \mathbb{O}(\mathbf{X})$, we may, in the unital case, replace the assumption of \mathbf{X} -purely largeness in Theorem 6.11 with the condition that the extension is \mathbf{X} -full. This is in general much easier to verify. In the non-unital case however, fullness of the extension will not be enough to guarantee absorption. This is the motivation for the following definition.

Definition 6.15. Let \mathfrak{A} be a lower semicontinuous \mathbf{X} - C^* -algebra and \mathfrak{D} be a unital \mathbf{X} - C^* -algebra such that $\mathfrak{D}(\mathbf{X}) = \mathfrak{D}$. An \mathbf{X} -equivariant $*$ -homomorphism $\phi : \mathfrak{A} \rightarrow \mathfrak{D}$ is said to be *unitisably full* (or unitisably \mathbf{X} -full) if the induced unital, \mathbf{X} -equivariant $*$ -homomorphism $\phi^\dagger : \mathfrak{A}^\dagger \rightarrow \mathfrak{D}$ is full.

An \mathbf{X} -equivariant extension of \mathfrak{A} by \mathfrak{B} is said to be *unitisably full* if the Busby map $\tau : \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$ is unitisably full.

Clearly a unitisably full $*$ -homomorphism is necessarily full. Moreover, if \mathfrak{A} is unital, then $\phi: \mathfrak{A} \rightarrow \mathfrak{D}$ is unitisably full if and only if ϕ is full and $1_{\mathfrak{D}} - \phi(1_{\mathfrak{A}})$ is full in \mathfrak{D} . In particular, a unital $*$ -homomorphism can never be unitisably full.

The following shows that the X -purely large extensions we are interested in are (unitisably) X -full.

Lemma 6.16. *Let X be a finite space, and let $\epsilon: 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of X - C^* -algebras. Suppose that \mathfrak{A} is a separable, lower semicontinuous X - C^* -algebra, and \mathfrak{B} is a stable, X - σ -unital, continuous X - C^* -algebra.*

If ϵ is X -purely large, then it is X -full. Moreover, if ϵ in addition absorbs the zero extension, then it is unitisably X -full.

Proof. If ϵ is X -purely large, then so is its sum with the zero extension. Also, ϵ is X -full if and only if $\epsilon \oplus 0$ is X -full. Thus it suffices to show that when ϵ absorbs the zero extension, then it is unitisably X -full, so we should show that the unitised extension ϵ^\dagger is X -full. Note that ϵ^\dagger is X -purely large by Proposition 4.17.

Let $\mathcal{C} := CP_{\text{r-nuc}}(X; \mathfrak{A}^\dagger, \mathfrak{B})$. By Corollary 6.7 there is a unital \mathcal{C} -absorbing $*$ -homomorphism $\Phi: \mathfrak{A}^\dagger \rightarrow \mathcal{M}(\mathfrak{B})$ weakly in \mathcal{C} which is X -full. Since $\mathcal{Q}(\mathfrak{B})(U) = \pi(\mathcal{M}(\mathfrak{B})(U))$ for all $U \in \mathbb{O}(X)$, it follows that $\pi \circ \Phi: \mathfrak{A}^\dagger \rightarrow \mathcal{Q}(\mathfrak{B})$ is X -full. Moreover, $\pi \circ \Phi$ is the Busby map of a trivial, unital \mathcal{C} -extension \mathfrak{f} , and thus ϵ^\dagger absorbs \mathfrak{f} . Since the Cuntz sum of an X -equivariant map with an X -full map is clearly X -full, it follows that ϵ^\dagger is an X -full extension. \square

If we assume that $\mathfrak{B}(U)$ has the corona factorisation property for every $U \in \mathbb{O}(X)$ then we also get a converse of the above lemma. We will need a few intermediate results.

Recall, that if \mathfrak{J} is a two-sided, closed ideal in \mathfrak{B} , then there is a canonical injective $*$ -homomorphism $\iota: \mathcal{M}(\mathfrak{B}, \mathfrak{J}) \hookrightarrow \mathcal{M}(\mathfrak{J})$. By the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & \mathcal{M}(\mathfrak{B}, \mathfrak{J}) & \longrightarrow & \mathcal{Q}(\mathfrak{B}, \mathfrak{J}) \longrightarrow 0 \\ & & \parallel & & \downarrow \iota & & \downarrow \bar{\iota} \\ 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & \mathcal{M}(\mathfrak{J}) & \xrightarrow{\pi} & \mathcal{Q}(\mathfrak{J}) \longrightarrow 0 \end{array}$$

which has exact rows, there is an induced $*$ -homomorphism $\bar{\iota}: \mathcal{Q}(\mathfrak{B}, \mathfrak{J}) \rightarrow \mathcal{Q}(\mathfrak{J})$. A diagram chase above shows that $\bar{\iota}$ is injective.

Lemma 6.17. *Let \mathfrak{B} be a stable, σ -unital C^* -algebra, and \mathfrak{J} be a two-sided, closed ideal in \mathfrak{B} which contains a full element. Then $\bar{\iota}(\mathcal{Q}(\mathfrak{B}, \mathfrak{J}))$ is a full hereditary C^* -subalgebra of $\mathcal{Q}(\mathfrak{J})$.*

Proof. It follows from Lemma 6.5 that $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J}))$ is a full hereditary C^* -subalgebra of $\mathcal{M}(\mathfrak{J})$. Since any $*$ -epimorphism maps full hereditary C^* -subalgebras onto full hereditary C^* -subalgebras, $\bar{\iota}(\mathcal{Q}(\mathfrak{B}, \mathfrak{J})) = \pi(\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J})))$ is a full hereditary C^* -subalgebra of $\mathcal{Q}(\mathfrak{J})$. \square

Lemma 6.18. *Let \mathfrak{B} be a stable σ -unital C^* -algebra, and let \mathfrak{J} be a σ -unital, two-sided, closed ideal in \mathfrak{B} . Let $m \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$. Then m is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$ if and only if $\pi(m)$ is full in $\mathcal{Q}(\mathfrak{B}, \mathfrak{J})$.*

Proof. One direction is trivial, so suppose that $\pi(m)$ is full in $\mathcal{Q}(\mathfrak{B}, \mathfrak{J})$. By Lemma 6.17, $\bar{\iota}(\pi(m))$ is full in $\mathcal{Q}(\mathfrak{J})$. Since $\pi(\iota(m)) = \bar{\iota}(\pi(m))$, it follows from [KN06b, Proposition 3.3]

(clearly one does not need the positivity requirement in the proposition), that $\iota(m)$ is full in $\mathcal{M}(\mathfrak{J})$. It follows from Lemma 6.5 that m is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$. \square

Lemma 6.19. *Let \mathfrak{B} be a stable σ -unital C^* -algebra, and let \mathfrak{J} be a σ -unital, two-sided, closed ideal in \mathfrak{B} . Then $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$ has a full, properly infinite projection.*

Proof. The Hilbert \mathfrak{B} -module $E := \mathfrak{J}$ is countably generated since $\mathbb{K}(E) \cong \mathfrak{J}$ is σ -unital. By Kasparov's stabilisation theorem and since $\mathcal{H}_{\mathfrak{B}} \cong \mathfrak{B}$, there is a projection $P' \in \mathbb{B}(\mathfrak{B}) \cong \mathcal{M}(\mathfrak{B})$ such that $P'\mathfrak{B} \cong E$. Let $\eta: \mathbb{B}(\mathfrak{B}) \xrightarrow{\cong} \mathcal{M}(\mathfrak{B})$ be the canonical isomorphism. It easily follows that $\overline{\mathfrak{B}\eta(P')\mathfrak{B}} = \mathfrak{J}$. It follows from Lemma 6.5 that any infinite repeat P of $\eta(P')$ is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$. Since P is an infinite repeat, and is non-zero, it is unitarily equivalent any Cuntz sum $P \oplus P$, and is thus properly infinite. \square

Proposition 6.20. *Let X be a finite space, and let $\mathfrak{e}: 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ be an extension of X - C^* -algebras. Suppose that \mathfrak{A} is a separable, lower semicontinuous X - C^* -algebra, and \mathfrak{B} is a stable, X - σ -unital, continuous X - C^* -algebra. Suppose that $\mathfrak{B}(U)$ has the corona factorisation property for each $U \in \mathbb{O}(X)$. If \mathfrak{e} is X -full, then it is X -purely large.*

Proof. Let $x \in \mathfrak{E}$, and let $U \in \mathbb{O}(X)$ be such that $p(x)$ is U -full. We must show that $\overline{x\mathfrak{B}x^*}$ contains a σ -unital, stable C^* -subalgebra \mathfrak{D} which is full in $\mathfrak{B}(U)$. Let y be a strictly positive contraction in $\mathfrak{E}(U)$, which exists since $0 \rightarrow \mathfrak{B}(U) \rightarrow \mathfrak{E}(U) \rightarrow \mathfrak{A}(U) \rightarrow 0$ is an extension with $\mathfrak{A}(U)$ and $\mathfrak{B}(U)$ σ -unital. Then $p(y)$ is strictly positive in $\mathfrak{A}(U)$. Since $p(x) \in \mathfrak{A}(U)$, $p(xy)$ generates the same two-sided, closed ideal as $p(x)$, which implies that $p(xy)$ is U -full.

Let $\sigma: \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ be the induced $*$ -homomorphism. By Proposition 5.20, σ is X -equivariant so $\sigma(xy) \in \mathcal{M}(\mathfrak{B})(U)$. Since the extension is X -full, $\pi(\sigma(xy)) = \tau(p(xy))$ is full in $\mathcal{Q}(\mathfrak{B})(U)$, so by Lemma 6.18 $\sigma(xy)$ is full in $\mathcal{M}(\mathfrak{B})(U)$. By Lemma 6.5, $\iota(\sigma(xy))$ is full in $\mathcal{M}(\mathfrak{B}(U))$ so by [KN06b, Lemma 3.2] there is a full, multiplier projection $P \in \mathcal{M}(\mathfrak{B}(U))$ such that

$$\mathfrak{D} := \overline{xy\mathfrak{B}(xy)^*} = \overline{\iota(\sigma(xy))\mathfrak{B}(U)\iota(\sigma(xy))^*} \cong P\mathfrak{B}(U)P.$$

Since $\mathfrak{B}(U)$ has the corona factorisation property it follows that \mathfrak{D} is stable, and it is clearly full in $\mathfrak{B}(U)$. Thus \mathfrak{e} is X -purely large since \mathfrak{D} is obviously σ -unital and a C^* -subalgebra of $\overline{x\mathfrak{B}x^*}$. \square

In order to get some results in the spirit of Voiculescu's Weyl-von Neumann type absorption theorem [Voi76], we will need the following definitions.

Definition 6.21. Let \mathfrak{D} be an X - C^* -algebra. We say that \mathfrak{C} is an X - C^* -subalgebra of \mathfrak{D} if \mathfrak{C} is an X - C^* -algebra such that the underlying C^* -algebra is a C^* -subalgebra of \mathfrak{D} , and $\mathfrak{C}(U) = \mathfrak{C} \cap \mathfrak{D}(U)$ for every $U \in \mathbb{O}(X)$.

Moreover, we say that \mathfrak{C} is a full X - C^* -subalgebra of \mathfrak{D} if \mathfrak{C} is an X - C^* -subalgebra of \mathfrak{D} and the inclusion map $\mathfrak{C} \hookrightarrow \mathfrak{D}$ is X -full.

We can now prove another main theorem. This theorem says that under the assumption of the corona factorisation property on each two-sided, closed ideal $\mathfrak{B}(U)$, we get much nicer absorption results.

Theorem 6.22. *Let X be a finite space and \mathfrak{B} be a stable, X - σ -unital, continuous X - C^* -algebra.² The following are equivalent.*

²I.e. a C^* -algebra over X , since X is finite.

- (i) $\mathfrak{B}(\mathcal{U})$ has the corona factorisation property for every $\mathcal{U} \in \mathcal{O}(\mathcal{X})$,
- (ii) for every separable, unital, lower semicontinuous \mathcal{X} - C^* -algebra \mathfrak{A} with $\mathfrak{B}(\mathcal{U}_{1_{\mathfrak{A}}}) = \mathfrak{B}$, it holds that every unital, \mathcal{X} -full extension of \mathfrak{A} by \mathfrak{B} , absorbs any trivial, unital weakly residually \mathcal{X} -nuclear extension of \mathfrak{A} by \mathfrak{B} ,
- (iii) for every separable, lower semicontinuous \mathcal{X} - C^* -algebra \mathfrak{A} , it holds that every unitisably \mathcal{X} -full extension of \mathfrak{A} by \mathfrak{B} , absorbs any trivial, weakly residually \mathcal{X} -nuclear extension of \mathfrak{A} by \mathfrak{B} ,
- (iv) for every separable, lower semicontinuous \mathcal{X} - C^* -algebra \mathfrak{A} , $KK_{\text{r-nuc}}^1(\mathcal{X}; \mathfrak{A}, \mathfrak{B})$ is (isomorphic to) the group of unitisably \mathcal{X} -full, weakly residually \mathcal{X} -nuclear extensions of \mathfrak{A} by \mathfrak{B} under multiplier unitary equivalence.
- (v) for any separable, unital \mathcal{X} - C^* -subalgebra $\mathfrak{C} \hookrightarrow \mathcal{M}(\mathfrak{B})$ for which $\mathfrak{C}/\mathfrak{B}$ is a full \mathcal{X} - C^* -subalgebra of $\mathcal{Q}(\mathfrak{B})$, it holds that any unital, weakly residually \mathcal{X} -nuclear map $\phi: \mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ for which $\phi(\mathfrak{C} \cap \mathfrak{B}) = 0$, is strongly asymptotically dominated by ι .
- (vi) for any separable, unital \mathcal{X} - C^* -subalgebra $\mathfrak{C} \hookrightarrow \mathcal{M}(\mathfrak{B})$ for which $\mathfrak{C}/\mathfrak{B}$ is a full \mathcal{X} - C^* -subalgebra of $\mathcal{Q}(\mathfrak{B})$, it holds for any unital, weakly residually \mathcal{X} -nuclear $*$ -homomorphism $\Phi: \mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ for which $\Phi(\mathfrak{C} \cap \mathfrak{B}) = 0$, that $\iota \oplus \Phi \sim_{as} \iota$.
- (vii) for any separable, unital, lower semicontinuous \mathcal{X} - C^* -algebra \mathfrak{A} with $\mathfrak{B}(\mathcal{U}_{1_{\mathfrak{A}}}) = \mathfrak{B}$, it holds that every unital, \mathcal{X} -full $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is unitally weakly residually \mathcal{X} -nuclearly absorbing.

Proof. (i) \Rightarrow (ii): Let \mathfrak{A} be as in (ii) and let $\mathfrak{e}: 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be a unital, \mathcal{X} -full extension. By Theorem 6.11 it suffices to show that \mathfrak{e} is \mathcal{X} -purely large. This follows from Proposition 6.20.

(ii) \Rightarrow (iii): Let \mathfrak{A} be as in (iii) and let $\mathfrak{e}: 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be a unitisably \mathcal{X} -full extension. By Proposition 4.17 it suffices to show that the unitised \mathcal{X} -equivariant extension \mathfrak{e}^\dagger absorbs any trivial, unital weakly residually \mathcal{X} -nuclear extension of \mathfrak{A}^\dagger by \mathfrak{B} . This follows from (ii) since \mathfrak{e}^\dagger is \mathcal{X} -full.

(iii) \Rightarrow (iv): It follows from [Kir00] that $KK_{\text{r-nuc}}^1(\mathcal{X}; \mathfrak{A}, \mathfrak{B})$ is canonically isomorphic to the group of absorbing $CP_{\text{r-nuc}}(\mathcal{X}; \mathfrak{A}, \mathfrak{B})$ -extensions under multiplier unitary equivalence. By absorbing we mean extensions which absorb any trivial $CP_{\text{r-nuc}}(\mathcal{X}; \mathfrak{A}, \mathfrak{B})$ -extensions. By (iii) these extensions are exactly the unitisably \mathcal{X} -full, weakly residually \mathcal{X} -nuclear extensions of \mathfrak{A} by \mathfrak{B} .

(iv) \Rightarrow (i): Suppose that $\mathcal{U} \in \mathcal{O}(\mathcal{X})$ such that $\mathfrak{B}(\mathcal{U})$ does not have the corona factorisation property, and let $\mathbb{C}_{\mathcal{U}}$ be the \mathcal{X} - C^* -algebra with underlying C^* -algebra \mathbb{C} , and action of \mathcal{X} given by

$$\mathbb{C}_{\mathcal{U}}(\mathcal{V}) = \begin{cases} \mathbb{C}, & \text{if } \mathcal{U} \subset \mathcal{V} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\mathbb{C}_{\mathcal{U}}$ is lower semicontinuous.

By Lemma 6.19, there there exists a multiplier projection P which is properly infinite and full in $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(\mathcal{U}))$. Now $P\mathfrak{B}P \cong \mathfrak{B}(\mathcal{U})$ and by [Kuc04, Lemma 11] $P\mathcal{M}(\mathfrak{B})P \cong \mathcal{M}(P\mathfrak{B}P)$ canonically. Since $\mathfrak{B}(\mathcal{U})$ does not have the corona factorisation property it follows that there is a projection Q_1 in $P\mathcal{M}(\mathfrak{B})P$ which is not properly infinite, but which is full in $P\mathcal{M}(\mathfrak{B})P$, and thus also full in $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(\mathcal{U}))$. Let $Q = Q_1 \oplus 0$ be a Cuntz sum in $\mathcal{M}(\mathfrak{B})$. Then Q is a full projection in $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(\mathcal{U}))$ which is not properly infinite. Moreover, since $0 \oplus 1 \leq 1 - Q$ it follows that $1 - Q$ is a full, properly infinite projection in $\mathcal{M}(\mathfrak{B})$.

Let ϵ denote the trivial extension of \mathbb{C}_U by \mathfrak{B} with X -equivariant splitting $\sigma: \mathbb{C}_U \rightarrow \mathcal{M}(\mathfrak{B})$ given by $\sigma(1) = Q$. Since Q is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$ and $1 - Q$ is full in $\mathcal{M}(\mathfrak{B})$ it easily follows that ϵ is unitisably X -full. Let $\mathcal{C} = CP_{r\text{-nuc}}(X; \mathbb{C}_U, \mathfrak{B})$. By Corollary 6.7 there exists a trivial \mathcal{C} -extension \mathfrak{f} which absorbs any trivial \mathcal{C} -extension, so this is X -purely large by Theorem 6.11 and thus unitisably X -full by Lemma 6.16. Any trivial \mathcal{C} -extension induces the zero element in $KK_{r\text{-nuc}}^1(X; \mathbb{C}_U, \mathfrak{B})$, so in particular $[\epsilon] = [\mathfrak{f}] = 0$. It suffices to show that ϵ does *not* absorb every trivial \mathcal{C} -extension, so assume for contradiction that ϵ *does* absorb every trivial \mathcal{C} -extension.

Consider the trivial extension ϵ_0 with X -equivariant splitting $\sigma_0: \mathbb{C}_U \rightarrow \mathcal{M}(\mathfrak{B})$ given by $\sigma_0(1) = P$, where P is as above. Since ϵ absorbs ϵ_0 we may find a unitary $u \in \mathcal{M}(\mathfrak{B})$ such that $u^*Qu - (Q \oplus P) \in \mathfrak{B}$. Note that $u^*Qu - (Q \oplus P) \in \mathfrak{B} \cap \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U)) = \mathfrak{B}(U)$. It is easily seen that $Q \oplus P \in \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$ and dominates the properly infinite projection $0 \oplus P$, which is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$. It follows that $Q \oplus P$ is properly infinite.

Let $\iota: \mathcal{M}(\mathfrak{B}) \rightarrow \mathcal{M}(\mathfrak{B}(U))$ be the canonical $*$ -homomorphism by considering each multiplier of \mathfrak{B} as a multiplier of $\mathfrak{B}(U)$. Note that the restriction of ι to $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$ is the canonical injective $*$ -homomorphism $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(U)) \hookrightarrow \mathcal{M}(\mathfrak{B}(U))$ considered in Section 6.1. By Lemma 6.5 $\iota(u^*Qu)$ and $\iota(Q \oplus P)$ are full in $\mathcal{M}(\mathfrak{B}(U))$. Since $Q \oplus P$ is properly infinite in $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$ and since the restriction of ι to $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$ is injective, it follows that $\iota(Q \oplus P)$ is properly infinite in $\mathcal{M}(\mathfrak{B}(U))$. Thus there is an isometry $v \in \mathcal{M}(\mathfrak{B}(U))$ with $vv^* = \iota(Q \oplus P)$. Since $\iota(u^*Qu) - \iota(Q \oplus P) = u^*Qu - (Q \oplus P) \in \mathfrak{B}(U)$ we get $b := v^*\iota(u^*Qu)v - 1_{\mathcal{M}(\mathfrak{B}(U))} \in \mathfrak{B}(U)$. Since $\mathfrak{B}(U)$ is stable we may find an isometry $w \in \mathcal{M}(\mathfrak{B}(U))$ such that $w^*bw < 1$. Hence

$$\|(\iota(u)vw)^*\iota(Q)(\iota(u)vw) - 1_{\mathcal{M}(\mathfrak{B}(U))}\| < 1.$$

This implies that $1_{\mathcal{M}(\mathfrak{B}(U))}$ is Murray–von Neumann subequivalent to $\iota(Q)$ and thus $\iota(Q)$ is properly infinite. Since $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{B}(U)))$ is a hereditary C^* -subalgebra of $\mathcal{M}(\mathfrak{B})$ by Lemma 6.4, this implies that Q is properly infinite, which is a contradiction to how we chose Q . Hence (iv) \Rightarrow (i).

(ii) \Rightarrow (v): Let \mathfrak{C} and ϕ be as in (v), and let $\mathfrak{A} := \mathfrak{C}/\mathfrak{B}$. Since the inclusion $\tau: \mathfrak{A} \hookrightarrow \mathcal{Q}(\mathfrak{B})$ is unital and X -full, it induces an X -full, unital extension of \mathfrak{A} by \mathfrak{B} , say ϵ , which has extension algebra $\mathfrak{E} = \mathfrak{C} + \mathfrak{B}$. Since \mathfrak{A} is an X - C^* -subalgebra of $\mathcal{Q}(\mathfrak{B})$ it follows that $U_{1_{\mathfrak{A}}} = U_{1_{\mathcal{Q}(\mathfrak{B})}}$, so $\mathfrak{B}(U_{1_{\mathfrak{A}}}) = \mathfrak{B}$. Hence $\mathcal{C} = CP_{r\text{-nuc}}(X; \mathfrak{A}, \mathfrak{B})$ is non-degenerate, and by Theorem 3.14 there exists a unital \mathcal{C} -absorbing $*$ -homomorphism $\Psi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ weakly in \mathcal{C} .

By (ii) the extension ϵ absorbs the extension with Busby map $\pi \circ \Psi: \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$. So there are \mathcal{O}_2 -isometries s_1, s_2 and a unitary u in $\mathcal{M}(\mathfrak{B})$ such that

$$u^*xu - (s_1xs_1^* + s_2\Psi(\pi(x))s_2^*) \in \mathfrak{B},$$

for all $x \in \mathfrak{C}$. Note that ϕ induces a map $\tilde{\phi}: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ which is weakly in \mathcal{C} , and thus $\tilde{\phi}$ is strongly asymptotically dominated by Ψ . Let $(v_t)_{t \in [1, \infty)}$ be a family of isometries implementing this strong approximate domination, and let $w_t := us_2v_t$. Then $\|w_t^*bw_t\| \rightarrow 0$ for all $b \in \mathfrak{B}$ and $w_t^*cw_t - \phi(c)$ is in \mathfrak{B} for all $c \in \mathfrak{C}$ and $t \in [1, \infty)$, and tends to zero as $t \rightarrow \infty$.

(v) \Rightarrow (vi): This follows from Theorem 3.4.

(ii) \Rightarrow (vii): Let \mathfrak{A} and Φ be as in (vii). Let $\Psi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ be a unital, weakly residually X -nuclear $*$ -homomorphism. Let Ψ_∞ be an infinite repeat of Ψ . Since Φ is X -full, it follows

that the extension with Busby map $\pi \circ \Phi$ is \mathbf{X} -full. Thus by (ii) it absorbs the extension with Busby map $\pi \circ \Psi_\infty$, so there is a unitary $U \in \mathcal{M}(\mathfrak{B})$ such that

$$U^*(\Phi(a) \oplus \Psi_\infty(a))U - \Phi(a) \in \mathfrak{B},$$

for all $a \in \mathfrak{A}$. By Theorem 3.4 Φ absorbs Ψ_∞ and thus also Ψ .

(vii) \Rightarrow (i): Suppose that $U \in \mathcal{O}(\mathbf{X})$ such that $\mathfrak{B}(U)$ does not have the corona factorisation property. Let $P, Q \in \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$ be the same projections as when proving (iv) \Rightarrow (i), and let $\mathfrak{C} := C^*(Q, 1_{\mathcal{M}(\mathfrak{B})})$ be the given \mathbf{X} - C^* -subalgebra of $\mathcal{M}(\mathfrak{B})$. There is an isomorphism of \mathbf{X} - C^* -algebras $\mathbb{C}_U^\dagger \rightarrow \mathfrak{C}$ given by $\mathbb{C}_U^\dagger = \mathbb{C}^2 \ni (\lambda, \mu) \mapsto \lambda Q + \mu(1 - Q)$. Thus, by how we chose Q , \mathfrak{C} is a full \mathbf{X} - C^* -subalgebra of $\mathcal{M}(\mathfrak{B})$. Moreover, \mathfrak{C} is lower semicontinuous.

Let $\Phi: \mathbb{C}_U^\dagger \rightarrow \mathcal{M}(\mathfrak{B})$ be given by $(\lambda, \mu) \mapsto \lambda P + \mu(1 - P)$. This is a unital, \mathbf{X} -equivariant $*$ -homomorphism which is obviously residually \mathbf{X} -nuclear. Statement (vii) would imply that there is a unitary $u \in \mathcal{M}(\mathfrak{B})$ such that $\|u^*Qu - P \oplus Q\| < 1$ which would imply that Q and $P \oplus Q$ are Murray–von Neumann equivalent. As seen in the proof of (iv) \Rightarrow (i), $P \oplus Q$ is properly infinite. Since Q was chosen not to be properly infinite, this implies that (vii) does not hold, and thus (vii) \Rightarrow (i).

(vi) \Rightarrow (i): The proof is identical to that of (vii) \Rightarrow (i), where one simply notes that $\mathfrak{C}/\mathfrak{B}$ is a full \mathbf{X} - C^* -subalgebra of $\mathcal{Q}(\mathfrak{B})$ and $\mathfrak{C} \cap \mathfrak{B} = 0$. \square

As a corollary more suited for classification, we obtain the following.

Corollary 6.23. *Let \mathbf{X} be a finite space and \mathfrak{B} be a stable, \mathbf{X} - σ -unital, continuous \mathbf{X} - C^* -algebra such that $\mathfrak{B}(U)$ has the corona factorisation property for each $U \in \mathcal{O}(\mathbf{X})$. Let \mathfrak{A} be a separable, nuclear, lower semicontinuous \mathbf{X} - C^* -algebra.*

Then $KK^1(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ is the group of unitisably \mathbf{X} -full extensions of \mathfrak{A} by \mathfrak{B} under multiplier unitary equivalence.

If, in addition, $\mathfrak{A}/\mathfrak{A}(U)$ is non-zero and non-unital for every $U \in \mathcal{O}(\mathbf{X}) \setminus \{\mathbf{X}\}$, then $KK^1(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ is the group of \mathbf{X} -full extensions of \mathfrak{A} by \mathfrak{B} under multiplier unitary equivalence.

Proof. We have that $KK^1(\mathbf{X}; \mathfrak{A}, \mathfrak{B}) = KK_{\text{r-nuc}}^1(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ when \mathfrak{A} is nuclear. Moreover, by Corollary 5.32 any \mathbf{X} -equivariant extension of \mathfrak{A} by \mathfrak{B} has an \mathbf{X} -equivariant contractive c.p. split, which is residually \mathbf{X} -nuclear since \mathfrak{A} is nuclear. Hence the first part follows from Theorem 6.22.

Suppose that $\mathfrak{A}/\mathfrak{A}(U)$ is non-zero and non-unital for every $U \in \mathcal{O}(\mathbf{X}) \setminus \{\mathbf{X}\}$. Recall, that $KK_{\text{r-nuc}}^1(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ is canonically isomorphic to the group of absorbing $CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ -extensions under multiplier unitary equivalence. Since any such extension absorbs an \mathbf{X} -full extension, it follows that any such extension is \mathbf{X} -full. By Proposition 6.20 any \mathbf{X} -full extension is \mathbf{X} -purely large and thus it absorbs any trivial $CP_{\text{r-nuc}}(\mathbf{X}; \mathfrak{A}, \mathfrak{B})$ -extension by Corollary 6.13, which finishes the proof. \square

7. A WEYL–VON NEUMANN THEOREM A LA KIRCHBERG

In this section we prove a Weyl–von Neumann type theorem, in the sense of Kirchberg in [Kir94], for C^* -algebras with a finite primitive ideal space. We also obtain a related result regarding extensions by such C^* -algebras.

Recall, that an \mathbf{X} - C^* -algebra \mathfrak{B} is called *tight* if the action $\mathcal{O}(\mathbf{X}) \rightarrow \mathbb{I}(\mathfrak{B})$ is a complete lattice isomorphism.

Definition 7.1. We say that an X - C^* -algebra \mathfrak{B} has a *tight corona algebra*, if the induced action of X on $\mathcal{Q}(\mathfrak{B})$ is tight.

The following lemma shows that if \mathfrak{B} is a stable and tight X - C^* -algebra, then the action $\mathbb{O}(X) \rightarrow \mathbb{I}(\mathcal{Q}(\mathfrak{B}))$ is a lattice embedding.

Lemma 7.2. *Let \mathfrak{B} be a stable C^* -algebra, and let \mathfrak{I} and \mathfrak{J} be two-sided, closed ideals in \mathfrak{B} . Then $\mathfrak{I} \subset \mathfrak{J}$ if and only if $\mathcal{Q}(\mathfrak{B}, \mathfrak{I}) \subset \mathcal{Q}(\mathfrak{B}, \mathfrak{J})$.*

Proof. One implication is trivial. Suppose that $\mathcal{Q}(\mathfrak{B}, \mathfrak{I}) \subset \mathcal{Q}(\mathfrak{B}, \mathfrak{J})$. Let $x \in \mathfrak{I}$, and let $t_1, t_2, \dots \in \mathcal{M}(\mathfrak{B})$ be isometries such that $\sum_{n=1}^{\infty} t_n t_n^*$ converges strictly to 1. Then

$$x_{\infty} = \sum_{n=1}^{\infty} t_n x t_n^* \in \mathcal{M}(\mathfrak{B}, \mathfrak{J}).$$

Since the inclusion $\mathcal{M}(\mathfrak{B}, \mathfrak{J}) \hookrightarrow \mathcal{M}(\mathfrak{B}, \mathfrak{J}) + \mathfrak{B}$ induces an isomorphism $\mathcal{M}(\mathfrak{B}, \mathfrak{J})/\mathfrak{J} \cong \mathcal{Q}(\mathfrak{B}, \mathfrak{J})$, and since $\pi(x_{\infty}) \in \mathcal{Q}(\mathfrak{B}, \mathfrak{J})$, there is a $y \in \mathfrak{J}$ such that $x_{\infty} + y \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$. Thus,

$$t_n^*(x_{\infty} + y)t_n = x + t_n^* y t_n \in \mathfrak{B} \cap \mathcal{M}(\mathfrak{B}, \mathfrak{J}) = \mathfrak{J}.$$

Since $y \in \mathfrak{B}$, it follows that $t_n^* y t_n \rightarrow 0$ and thus $x \in \mathfrak{J}$. This implies that $\mathfrak{I} \subset \mathfrak{J}$. \square

Definition 7.3. Let \mathfrak{A} and \mathfrak{B} be X - C^* -algebra with \mathfrak{A} lower semicontinuous. A c.p. map $\phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ is called *weakly X -full* if

$$\overline{\mathfrak{B}\phi(a)\mathfrak{B}} = \mathfrak{B}(U_a)$$

for every $a \in \mathfrak{A}$.

Example 7.4. If \mathfrak{B} is lower semicontinuous, and $\mathfrak{A} \subset \mathcal{M}(\mathfrak{B})$ is an X - C^* -subalgebra, then the inclusion is weakly X -full.

Kirchberg proved the following theorem in [Kir94] in the case where X is the one-point space. That conditions (i) and (ii) below are equivalent in the one-point space case, was originally proven by Lin in [Lin91]. This was also proven by Rørdam in [Rør91] in the case where \mathfrak{B} was assumed to contain a full projection.

Theorem 7.5. *Let X be a finite space and let \mathfrak{B} be a stable, X - σ -unital, tight X - C^* -algebra. Then the following are equivalent.*

- (i) every simple subquotient of \mathfrak{B} is purely infinite or isomorphic to \mathbb{K} ,
- (ii) \mathfrak{B} has a tight corona algebra,
- (iii) for every unital, separable X - C^* -subalgebra $\iota: \mathfrak{C} \hookrightarrow \mathcal{M}(\mathfrak{B})$ it holds that every unital, weakly residually X -nuclear map $\phi: \mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ such that $\phi(\mathfrak{C} \cap \mathfrak{B}) = 0$, is approximately dominated by ι ,
- (iv) for every unital, separable X - C^* -subalgebra $\iota: \mathfrak{C} \hookrightarrow \mathcal{M}(\mathfrak{B})$ it holds for every unital, weakly residually X -nuclear $*$ -homomorphism $\Phi: \mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ such that $\Phi(\mathfrak{C} \cap \mathfrak{B}) = 0$, that $\iota \oplus \Phi \sim_{as} \iota$,
- (v) for every unital, separable, lower semicontinuous X - C^* -algebra \mathfrak{A} , satisfying that $\mathfrak{B}(U_{1_{\mathfrak{A}}}) = \mathfrak{B}$, it holds that any weakly X -full $*$ -homomorphism $\Phi: \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B})$ for which $\Phi(\mathfrak{A}) \cap \mathfrak{B} = 0$, is weakly residually X -nuclearly absorbing.

Proof. (i) \Rightarrow (ii): Suppose that $\mathfrak{I}, \mathfrak{J}$ are two-sided, closed ideals in \mathfrak{B} . By Lemma 7.2, $\mathfrak{I} \subset \mathfrak{J}$ if and only if $\mathcal{Q}(\mathfrak{B}, \mathfrak{I}) \subset \mathcal{Q}(\mathfrak{B}, \mathfrak{J})$. Hence, if the map $\mathbb{I}(\mathfrak{B}) \ni \mathfrak{I} \mapsto \mathcal{Q}(\mathfrak{B}, \mathfrak{I}) \in \mathbb{I}(\mathcal{Q}(\mathfrak{B}))$ is a one-to-one correspondence, then it is a lattice bijection, and thus \mathfrak{B} has a tight corona algebra. That the map is injective follows from Lemma 7.2 since \mathfrak{B} is stable. To show that it is surjective it suffices to show that $|\text{Prim } \mathfrak{B}| = |\text{Prim } \mathcal{Q}(\mathfrak{B})|$, which we will prove by induction. Note that this is equivalent to $2 \cdot |\text{Prim } \mathfrak{B}| = |\text{Prim } \mathcal{M}(\mathfrak{B})|$. If \mathfrak{B} is simple the result follows from [Rør91]. For a general \mathfrak{B} with $|\text{Prim } \mathfrak{B}|$ finite, fix a maximal, two-sided, closed ideal \mathfrak{I} in \mathfrak{B} . Note that $|\text{Prim } \mathfrak{I}| = |\text{Prim } \mathfrak{B}| - 1$. It follows from Lemma 6.5 that $\mathcal{M}(\mathfrak{B}, \mathfrak{I})$ embeds canonically as a full hereditary C^* -subalgebra of $\mathcal{M}(\mathfrak{I})$. Because of the short exact sequence

$$0 \rightarrow \mathcal{M}(\mathfrak{B}, \mathfrak{I}) \rightarrow \mathcal{M}(\mathfrak{B}) \rightarrow \mathcal{M}(\mathfrak{B}/\mathfrak{I}) \rightarrow 0,$$

it follows that $|\text{Prim } \mathcal{M}(\mathfrak{B})| = |\text{Prim } \mathcal{M}(\mathfrak{B}, \mathfrak{I})| + |\text{Prim } \mathcal{M}(\mathfrak{B}/\mathfrak{I})| = |\text{Prim } \mathcal{M}(\mathfrak{I})| + 2$. By an induction argument it follows that $|\text{Prim } \mathcal{M}(\mathfrak{B})| = 2 \cdot (|\text{Prim } \mathfrak{B}| - 1) + 2 = |\text{Prim } \mathfrak{B}|$ and thus $|\text{Prim } \mathfrak{B}| = |\text{Prim } \mathcal{Q}(\mathfrak{B})|$.

(ii) \Rightarrow (i): Again we show the result by induction on $|\text{Prim } \mathfrak{B}|$. If \mathfrak{B} is simple, then $\mathcal{Q}(\mathfrak{B})$ is simple by (ii), so by [Lin91, Theorem 3.8] \mathfrak{B} is either purely infinite or \mathbb{K} . Let $n \in \mathbb{N}$ and suppose that for any space X with $|X| < n$, we have (ii) \Rightarrow (i). Suppose $|X| = n$. For a general \mathfrak{B} as in the theorem, let $\mathfrak{I} \subset \mathfrak{J} \subset \mathfrak{B}$ be two-sided, closed ideals such that $\mathfrak{J}/\mathfrak{I}$ is simple. Suppose that $\mathfrak{I} \neq \mathfrak{B}$. Since \mathfrak{B} is tight, there is a proper open subset Y of X , such that $\mathfrak{B}(Y) = \mathfrak{I}$. Then \mathfrak{I} and $\mathcal{Q}(\mathfrak{B}, \mathfrak{I})$ get an induced Y - C^* -algebra structure by restricting the action of X . Clearly \mathfrak{I} and $\mathcal{Q}(\mathfrak{B}, \mathfrak{I})$ are tight Y - C^* -algebra and in particular $|\mathbb{O}(Y)| = |\mathbb{I}(\mathcal{Q}(\mathfrak{B}, \mathfrak{I}))|$. By Lemma 6.17, $\mathcal{Q}(\mathfrak{B}, \mathfrak{I})$ is a full hereditary C^* -subalgebra $\mathcal{Q}(\mathfrak{I})$, and thus $\mathbb{I}(\mathcal{Q}(\mathfrak{B}, \mathfrak{I})) \cong \mathbb{I}(\mathcal{Q}(\mathfrak{I}))$. Hence $|\mathbb{O}(Y)| = |\mathbb{I}(\mathcal{Q}(\mathfrak{I}))|$. Since the map

$$\mathbb{O}(Y) \ni U \mapsto \mathcal{Q}(\mathfrak{I}, \mathfrak{I}(U)) \in \mathbb{I}(\mathcal{Q}(\mathfrak{I}))$$

is an injective order embedding by Lemma 7.2, and since $|\mathbb{O}(Y)| = |\mathbb{I}(\mathcal{Q}(\mathfrak{I}))|$, it follows that this action is tight. Hence \mathfrak{I} has a tight corona algebra. Since $\mathfrak{I} \neq \mathfrak{B}$, it follows that $|Y| < |X|$, and by our induction hypothesis it follows that every simple subquotient of \mathfrak{I} , and thus $\mathfrak{I}/\mathfrak{I}$, is either purely infinite or \mathbb{K} .

It remains to show the case when $\mathfrak{I} = \mathfrak{B}$. We have a short exact sequence $0 \rightarrow \mathcal{Q}(\mathfrak{B}, \mathfrak{I}) \rightarrow \mathcal{Q}(\mathfrak{B}) \rightarrow \mathcal{Q}(\mathfrak{B}/\mathfrak{I}) \rightarrow 0$, and by Lemma 6.17, $|\text{Prim } \mathcal{Q}(\mathfrak{B}, \mathfrak{I})| = |\text{Prim } \mathcal{Q}(\mathfrak{I})| = |\text{Prim } \mathfrak{B}| - 1$. Thus

$$|\text{Prim } \mathfrak{B}| = |\text{Prim } \mathcal{Q}(\mathfrak{B})| = |\text{Prim } \mathfrak{B}| - 1 + |\text{Prim } \mathcal{Q}(\mathfrak{B}/\mathfrak{I})|$$

so $\mathcal{Q}(\mathfrak{B}/\mathfrak{I})$ is simple. It follows from the case $n = 1$ that $\mathfrak{B}/\mathfrak{I}$ is purely infinite or \mathbb{K} .

(i) \Rightarrow (iii): It follows from [Rør02, Proposition 4.1.1] that any simple, stable, purely infinite C^* -algebra has the corona factorisation property, and clearly \mathbb{K} has the corona factorisation property. Hence, by [KN06a, Theorem 3.1] and the remark following that theorem, it follows that $\mathfrak{B}(U)$ has the corona factorisation property for every $U \in \mathbb{O}(X)$. Let \mathfrak{C} and ϕ be as in condition (iii). By Theorem 6.22 it suffices to show that $\mathfrak{C}/\mathfrak{B} \subset \mathcal{Q}(\mathfrak{B})$ is full X - C^* -subalgebra. Since any X - C^* -subalgebra of a tight X - C^* -algebra is full, this follows from (i) \Leftrightarrow (ii).

(iii) \Rightarrow (iv): This follows from Theorem 3.4.

(iv) \Rightarrow (i): Suppose that (i) does *not* hold. Then there are two-sided, closed ideals $\mathfrak{I} \subset \mathfrak{J}$ in \mathfrak{B} such that $\mathfrak{J}/\mathfrak{I}$ is simple, but is neither purely infinite nor \mathbb{K} . We will prove something

slightly more general than is needed for the proof, so that the proof for $(v) \Rightarrow (i)$ carries over with the same proof. We need the following.

Claim: there is a positive element $x \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ which is *not* full, such that $C^*(x) \cap \mathfrak{B} = 0$.

Let us first finish the proof of $(iv) \Rightarrow (i)$ under the assumption of this claim, and then prove the claim afterwards.

Let $h \in \mathfrak{J}$ be strictly positive. By replacing x with a Cuntz sum $x \oplus h \oplus 0$, we may furthermore assume that $\mathfrak{B}x\mathfrak{B} = \mathfrak{J}$ and $C^*(1, x) \cap \mathfrak{B} = 0$. Let $\mathfrak{C} = C^*(1, x)$ which we consider as an \mathbf{X} - C^* -subalgebra of $\mathcal{M}(\mathfrak{B})$, and let $\Phi: \mathfrak{C} \rightarrow \mathcal{M}(\mathfrak{B})$ be the infinite repeat of the inclusion ι . Since ι is \mathbf{X} -equivariant (by definition) so is Φ , and since \mathfrak{C} is nuclear, Φ is weakly residually \mathbf{X} -nuclear. Also $\Phi(\mathfrak{C} \cap \mathfrak{B}) = \Phi(0) = 0$. Suppose that (iv) holds. Then $\iota \oplus \Phi \sim_{as} \iota$, so in particular x and $x \oplus \Phi(x)$ generate the same two-sided, closed ideal in $\mathcal{M}(\mathfrak{B})$. However, since x is not full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$, and $x \oplus \Phi(x)$ is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$ by Lemma 6.5, we get a contradiction, so (iv) can not hold. Thus it remains to prove the above claim.

The proof of the claim is easy in the case where $\mathfrak{J} \neq 0$. By Lemma 6.19, we may find a (necessarily non-zero) projection P which is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$. Let (t_n) be a sequence of isometries in $\mathcal{M}(\mathfrak{B})$ such that $\sum t_n t_n^*$ converges strictly to 1, and let (λ_n) be a dense sequence in $[0, 1]$. Let $y = \sum_{n=1}^{\infty} \lambda_n t_n P t_n^*$. Then y is a positive element in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$ with $\text{spec}(y) = [0, 1]$ such that for any non-zero $h \in C_0((0, 1])$, $h(y)$ is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$.

Let $\tau: \mathcal{Q}(\mathfrak{B}, \mathfrak{J}) \rightarrow \mathcal{Q}(\mathfrak{J})$ be the embedding from Lemma 6.17. Clearly $\tau(\mathcal{Q}(\mathfrak{B}, \mathfrak{J}))$ is full in $\mathcal{Q}(\mathfrak{J}, \mathfrak{J})$. We get that $\mathcal{Q}(\mathfrak{J})/\mathcal{Q}(\mathfrak{J}, \mathfrak{J}) \cong \mathcal{Q}(\mathfrak{J}/\mathfrak{J})$ is non-simple by the equivalence of (i) and (ii) . Since $\tau(\mathcal{Q}(\mathfrak{B}, \mathfrak{J}))$ is a full, hereditary C^* -subalgebra of $\mathcal{Q}(\mathfrak{J})$ by Lemma 6.18, it follows that $\mathcal{Q}(\mathfrak{B}, \mathfrak{J})/\mathcal{Q}(\mathfrak{B}, \mathfrak{J}) \cong \mathcal{M}(\mathfrak{B}, \mathfrak{J})/(\mathcal{M}(\mathfrak{B}, \mathfrak{J}) + \mathfrak{J})$ is not simple. Thus we may find a positive contraction $x' \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ which is not full, and which is not in $\mathcal{M}(\mathfrak{B}, \mathfrak{J}) + \mathfrak{J}$. Let x be a Cuntz sum $x' \oplus y$. We clearly have that $x \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ is not full, and that $C^*(x) \cap \mathfrak{B} = 0$ by how we chose y . This finishes the proof of the claim in the case where $\mathfrak{J} \neq 0$.

Now suppose that $\mathfrak{J} = 0$ and thus \mathfrak{J} is simple. Since \mathfrak{J} is stable, σ -unital and is neither purely infinite nor \mathbb{K} , $\mathcal{M}(\mathfrak{J})$ contains a non-trivial ideal by [Lin91]. Thus we may find a positive contraction $m \in \mathcal{M}(\mathfrak{J})$ which is not in \mathfrak{J} and which is not full in $\mathcal{M}(\mathfrak{J})$. Let (y_n) be a countable approximate identity in \mathfrak{J} such that $y_n y_{n+1} = y_{n+1} y_n = y_n$. By passing to a subsequence we may assume that $\|(1 - y_n) m y_k\| < 2^{-(n+k)}$ for $k < n$. Let $d_n = y_n - y_{n-1}$ where we define $y_0 = 0$. Exactly as in the proof of Lemma 5.17 part (c), it follows that

$$m \in \overline{\mathcal{M}(\mathfrak{J}) \left(\sum_{n=1}^{\infty} d_n m d_n \right) \mathcal{M}(\mathfrak{J}) + \mathfrak{J}}.$$

Note that all our sums converge in the strict topology. Let $m_0 := \sum_{n=1}^{\infty} d_{2n} m d_{2n}$ and $m_1 := \sum_{n=1}^{\infty} d_{2n-1} m d_{2n-1}$, and note that $\sum_{n=1}^{\infty} d_n m d_n = m_0 + m_1$. Since

$$\left(\sum_{k=1}^{\infty} d_{2k} \right) m \left(\sum_{l=1}^{\infty} d_{2l} \right) = m_0 + \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} (d_{2k} m d_{2l} + d_{2l} m d_{2k})$$

one may use the norm estimate (exactly as in the proof of Lemma 5.17 part (c)) to show that the latter sum above is in \mathfrak{J} . Thus m_0 , and similarly m_1 , is in $\overline{\mathcal{M}(\mathfrak{J}) m \mathcal{M}(\mathfrak{J})} + \mathfrak{J}$. It follows that $m_0 + m_1 + \mathfrak{J}$ generates the same two-sided, closed ideal in $\mathcal{Q}(\mathfrak{J})$ as $m + \mathfrak{J}$. Since

m_0 and m_1 are positive, it follows that neither m_0 nor m_1 is full in $\mathcal{M}(\mathfrak{J})$, and at least one of m_0 and m_1 is not in \mathfrak{J} . Assume without loss of generality that m_0 is not in \mathfrak{J} .

Since $d_{2n}md_{2n}$ and $d_{2k}md_{2k}$ are positive and orthogonal for $k \neq n$, it follows that the sequence $(\|d_{2n}md_{2n}\|)_{n=1}^\infty$ does not tend to zero, for otherwise $m_0 \in \mathfrak{J}$. Hence we may find $0 < \epsilon < 1$ and an increasing sequence $(n(k))_{k=1}^\infty$ of positive integers, such that $\|d_{2n(k)}md_{2n(k)}\| > \epsilon$ for all k . Let $f, g: [0, 1] \rightarrow [0, 1]$ be the continuous functions given by

$$f(t) = \begin{cases} 0, & \text{for } t = 0 \\ \text{affine}, & \text{for } 0 \leq t \leq \epsilon/2 \\ 1, & \text{for } \epsilon/2 \leq t \leq 1 \end{cases}, \quad g(t) = \begin{cases} 0, & \text{for } 0 \leq t \leq \epsilon/2 \\ \text{affine}, & \text{for } \epsilon/2 \leq t \leq 1 - \epsilon/2 \\ 1, & \text{for } 1 - \epsilon/2 \leq t \leq 1. \end{cases}$$

Then $f(m_0) = \sum_{n=1}^\infty f(d_{2n}md_{2n})$, and $fg = gf = g$.

It is well-known that if \mathfrak{D} is a simple, σ -unital C^* -algebra not stably isomorphic to \mathbb{K} , then \mathfrak{D} contains a positive element d with $\text{spec}(d) = [0, 1]$. Since $\epsilon < \|d_{2n(k)}md_{2n(k)}\| \leq 1$ for every k , it follows that $g(d_{2n(k)}md_{2n(k)})$ is non-zero. Since \mathfrak{J} is simple, σ -unital and not (stably) isomorphic to \mathbb{K} , we may for each k find a positive element $x_k \in \overline{g(d_{2n(k)}md_{2n(k)})\mathfrak{J}g(d_{2n(k)}md_{2n(k)})}$ such that $\text{spec}(x_k) = [0, 1]$. Since $fg = gf = g$ we have that $f(m_0)x_k = x_k f(m_0) = x_k$. Clearly $\sum_{k=1}^\infty x_k$ converges strictly to an element x' , such that $f(m_0)x' = x' f(m_0) = x'$. Since $f(m_0)$ is in the C^* -algebra generated by m_0 , and is thus not full in $\mathcal{M}(\mathfrak{J})$, it follows that x' is not full in $\mathcal{M}(\mathfrak{J})$. Since x_k and x_l are positive orthogonal elements, it follows that if $h \in C_0((0, 1])$ then $h(x') = \sum_{k=1}^\infty h(x_k)$. Let $h \in C_0((0, 1])$ be non-zero. Since $\|h(x_k)\| = \|h\| > 0$, it follows that $h(x')$ is not in \mathfrak{J} . Thus $C^*(x') \cap \mathfrak{J} = 0$.

By Kasparov's stabilisation theorem, we may find a projection $P \in \mathcal{M}(\mathfrak{B})$ such that $P\mathfrak{B}P$ is full in \mathfrak{J} . By replacing P with an infinite repeat, we may assume that $P\mathfrak{B}P \cong \mathfrak{J}$, and $P\mathcal{M}(\mathfrak{B})P \cong \mathcal{M}(\mathfrak{J})$ is a corner which is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$ by Lemma 6.5. Let $x \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ be the element corresponding to $x' \in \mathcal{M}(\mathfrak{J})$ by the above isomorphism. Since $P\mathcal{M}(\mathfrak{B})P$ is a full, hereditary C^* -subalgebra of $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$, and x is not full in $P\mathcal{M}(\mathfrak{B})P$, it follows that x is not full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$. Also, since P is a unit for $C^*(x)$, it follows that $C^*(x) \cap \mathfrak{B} = C^*(x) \cap P\mathfrak{B}P = 0$. This proves the claim in the case where $\mathfrak{J} = 0$, and thus finishes the proof of $(iv) \Rightarrow (i)$.

$(i) \Rightarrow (v)$: Let \mathfrak{A} and Φ be as in (v) . By the equivalence of (i) and (ii) it easily follows that $\pi \circ \Phi$ is \mathbf{X} -full. By Lemma 6.18 it follows that Φ is \mathbf{X} -full. Thus, as in $(i) \Rightarrow (iii)$, we obtain (v) by Theorem 6.22.

$(v) \Rightarrow (i)$: The argument is identical to that of $(iv) \Rightarrow (i)$ with $\mathfrak{C} = \mathfrak{A}$. \square

We also want results for extensions by \mathbf{X} - C^* -algebras with a tight corona algebra. We do this by introducing the following actions.

Remark 7.6 (The induced action of an extension). Let $\epsilon: 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras, and let $\sigma: \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ and $\tau: \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$ be the induced $*$ -homomorphisms. Suppose that \mathfrak{B} is an \mathbf{X} - C^* -algebra. Then ϵ induces an \mathbf{X} - C^* -algebra structure on both \mathfrak{A} and \mathfrak{E} by

$$\begin{aligned} \mathfrak{A}(\mathbf{U}) &= \tau^{-1}(\mathcal{Q}(\mathfrak{B}, \mathfrak{B}(\mathbf{U}))) \\ \mathfrak{E}(\mathbf{U}) &= \sigma^{-1}(\mathcal{M}(\mathfrak{B}, \mathfrak{B}(\mathbf{U}))) \end{aligned}$$

for $\mathbf{U} \in \mathbb{O}(\mathbf{X})$.

There is an easy way of determining $\mathfrak{E}(\mathbf{U})$. Let $\mathbf{U} \in \mathbb{O}(\mathbf{X})$. We clearly have

$$\mathfrak{E}(\mathbf{U}) = \overline{\sum_{\mathfrak{J} \triangleleft \mathfrak{E}, \mathfrak{J} \cap \mathfrak{B} \subset \mathfrak{B}(\mathbf{U})} \mathfrak{J}},$$

so $\mathfrak{E}(\mathbf{U})$ is the *unique largest two-sided, closed ideal such that $\mathfrak{E}(\mathbf{U}) \cap \mathfrak{B} = \mathfrak{B}(\mathbf{U})$* . Thus, if \mathfrak{J} is any two-sided, closed ideal in \mathfrak{E} such that $\mathfrak{J} \cap \mathfrak{B} \subset \mathfrak{B}(\mathbf{U})$, then $\mathfrak{J} \subset \mathfrak{E}(\mathbf{U})$.

This description can often be used to determine the action on \mathfrak{E} , and since $\mathfrak{A}(\mathbf{U}) = p(\mathfrak{E}(\mathbf{U}))$, we may use this description to determine $\mathfrak{A}(\mathbf{U})$.

In this paper, we will only consider the induced action when \mathfrak{B} is a tight \mathbf{X} - C^* -algebra.

Remark 7.7. Let $\mathfrak{e} : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras. It follows from Lemma 5.17 that the action of $\text{Prim } \mathfrak{B}$ on \mathfrak{A} is finitely lower semicontinuous and that the action on \mathfrak{E} is lower semicontinuous.

Note that if $\text{Prim } \mathfrak{B}$ is finite, then by Corollary 5.31 this extension will have a $\text{Prim } \mathfrak{B}$ -equivariant c.p. split if \mathfrak{A} is nuclear and separable.

Remark 7.8. Recall, that when $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ is an extension of C^* -algebras, then $\text{Prim } \mathfrak{B}$ embeds canonically as an open subset of $\text{Prim } \mathfrak{E}$, such that $\text{Prim } \mathfrak{E} \setminus \text{Prim } \mathfrak{B}$ is canonically homeomorphic to $\text{Prim } \mathfrak{A}$. Thus $\text{Prim } \mathfrak{E}$ as a *set* may be canonically identified with the disjoint union $\text{Prim } \mathfrak{A} \sqcup \text{Prim } \mathfrak{B}$.

If the induced $\text{Prim } \mathfrak{B}$ - C^* -algebra structure on \mathfrak{A} is lower semicontinuous, then we may retrieve the topology on $\text{Prim } \mathfrak{E}$ from the $\text{Prim } \mathfrak{B}$ - C^* -algebra structure on \mathfrak{A} .

In fact, since $\text{Prim } \mathfrak{E} = \text{Prim } \mathfrak{A} \sqcup \text{Prim } \mathfrak{B}$ any open set will be of the form $\mathbf{U} \sqcup \mathbf{V}$, where $\mathbf{U} \in \mathbb{O}(\text{Prim } \mathfrak{A})$ and $\mathbf{V} \in \mathbb{O}(\text{Prim } \mathfrak{B})$. Let $\mathfrak{J}_{\mathbf{U}}$ be the closed, two-sided ideal in \mathfrak{A} corresponding to \mathbf{U} . If the action of $\text{Prim } \mathfrak{B}$ on \mathfrak{A} is lower semicontinuous, then there is a unique smallest open set $\mathbf{W}_{\mathbf{U}}$ in $\text{Prim } \mathfrak{B}$ such that $\mathfrak{J}_{\mathbf{U}} \subset \mathfrak{A}(\mathbf{W}_{\mathbf{U}})$. It is easily verified that $\mathbf{U} \sqcup \mathbf{V}$ is open in $\text{Prim } \mathfrak{E}$ if and only if $\mathbf{W}_{\mathbf{U}} \subset \mathbf{V}$.

We will provide two examples. The first example shows that the induced action on the extension algebra and the quotient need not be finitely upper semicontinuous. The second example shows that the induced action on the quotient is not necessarily lower semicontinuous.

Example 7.9. Let \mathfrak{E} be a C^* -algebra containing exactly three non-trivial, two-sided, closed ideals \mathfrak{B}_1 , \mathfrak{B}_2 and $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$ such that $\mathfrak{B}_1 \cap \mathfrak{B}_2 = 0$. Let $\mathfrak{A} = \mathfrak{E}/\mathfrak{B}$ and consider the extension $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$. Since $\mathfrak{B} \cong \mathfrak{B}_1 \oplus \mathfrak{B}_2$, the primitive ideal space \mathbf{X} of \mathfrak{B} is homeomorphic to the two-point discrete space $\{1, 2\}$ with open sets $\mathbb{O}(\mathbf{X}) = \{\emptyset, \{1\}, \{2\}, \mathbf{X}\}$. The induced actions on \mathfrak{A} and \mathfrak{E} can be determined as in Remark 7.6 and are

$$\begin{aligned} \mathfrak{E}(\emptyset) &= 0, & \mathfrak{E}(\{1\}) &= \mathfrak{B}_1, & \mathfrak{E}(\{2\}) &= \mathfrak{B}_2, & \mathfrak{E}(\mathbf{X}) &= \mathfrak{E} \\ \mathfrak{A}(\emptyset) &= 0, & \mathfrak{A}(\{1\}) &= 0, & \mathfrak{A}(\{2\}) &= 0, & \mathfrak{A}(\mathbf{X}) &= \mathfrak{A}. \end{aligned}$$

These actions are clearly not finitely upper semicontinuous, since this would imply that $\mathfrak{A}(\{1\}) + \mathfrak{A}(\{2\}) = \mathfrak{A}$ and $\mathfrak{E}(\{1\}) + \mathfrak{E}(\{2\}) = \mathfrak{E}$.

Example 7.10. Let \mathfrak{e} be the extension

$$0 \rightarrow C_0((0, 1]) \rightarrow C([0, 1]) \rightarrow \mathbb{C} \rightarrow 0.$$

Then the action of $(0, 1]$ on \mathbb{C} induced by \mathfrak{e} is given by

$$\mathbb{C}(U) = \begin{cases} 0, & \text{if } U \subset [\epsilon, 1] \text{ for some } \epsilon \in (0, 1] \\ \mathbb{C}, & \text{otherwise.} \end{cases}$$

Since $\bigcap_{n \in \mathbb{N}} \mathbb{C}((0, 1/n)) = \mathbb{C}$, it follows that this action is not lower semicontinuous.

In classification theory it is often desirable to classify tight X - C^* -algebras. If $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \xrightarrow{p} \mathfrak{A} \rightarrow 0$ is an extension of C^* -algebras, then there are induced actions of $\text{Prim } \mathfrak{E}$ on \mathfrak{A} and \mathfrak{B} . These are simply given by $\mathfrak{A}(U) = p(\mathfrak{E}(U))$ and $\mathfrak{B}(U) = \mathfrak{E}(U) \cap \mathfrak{B}$, for $U \in \mathbb{O}(\text{Prim } \mathfrak{E})$.

Warning 7.11. It is important to specify which of these two actions we are using on \mathfrak{A} . E.g. since $\mathbb{O}(\text{Prim } \mathfrak{B}) \subset \mathbb{O}(\text{Prim } \mathfrak{E})$, then $\mathfrak{A}(\text{Prim } \mathfrak{B}) = \mathfrak{A}$ if we use the $\text{Prim } \mathfrak{B}$ action, but $\mathfrak{A}(\text{Prim } \mathfrak{B}) = 0$ if we use the $\text{Prim } \mathfrak{E}$ action.

However, $\mathfrak{B}(U)$ for $U \in \mathbb{O}(\text{Prim } \mathfrak{B})$ does not depend on the choice of action.

The following lemma says that considering this action, or the action induced by the ideal \mathfrak{B} as above, is essentially the same.

Lemma 7.12. *Let $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras. Then*

$$CP(\text{Prim } \mathfrak{B}; \mathfrak{A}, \mathfrak{B}) = CP(\text{Prim } \mathfrak{E}; \mathfrak{A}, \mathfrak{B}).$$

Proof. To avoid confusion we write $\mathfrak{A}_{\mathfrak{B}}$ when we mean \mathfrak{A} with the $\text{Prim } \mathfrak{B}$ action, and $\mathfrak{A}_{\mathfrak{E}}$ when we mean \mathfrak{A} with the $\text{Prim } \mathfrak{E}$ action.

Let $\phi \in CP(\text{Prim } \mathfrak{E}; \mathfrak{A}, \mathfrak{B})$ and $U \in \mathbb{O}(\text{Prim } \mathfrak{B}) \subset \mathbb{O}(\text{Prim } \mathfrak{E})$. Let $U_{\mathfrak{B}} \in \mathbb{O}(\text{Prim } \mathfrak{E})$ such that $\mathfrak{E}(U_{\mathfrak{B}}) = \mathfrak{B}$, and define

$$V_U = \bigcup_{V \in \mathbb{O}(\text{Prim } \mathfrak{E}), V \cap U_{\mathfrak{B}} = U \cap U_{\mathfrak{B}}} V.$$

Then V_U is the largest open subset of $\text{Prim } \mathfrak{E}$ such that $V_U \cap U_{\mathfrak{B}} = U \cap U_{\mathfrak{B}}$.

Let $\sigma: \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ and $\tau: \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$ be the canonical maps, and let $a \in \mathfrak{A}_{\mathfrak{B}}(U)$. Then a lifts to $x \in \mathfrak{E}$ such that $\sigma(x) \in \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$. As $\sigma^{-1}(\mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))) \cap \mathfrak{B} = \mathfrak{B}(U) = \mathfrak{E}(U) \cap \mathfrak{B}$, it follows that $x \in \mathfrak{E}(V_U)$. Hence $a \in \mathfrak{A}_{\mathfrak{E}}(V_U)$ and thus $\phi(a) \in \mathfrak{B}(V_U) = \mathfrak{B}(U)$, which implies that $\phi \in CP(\text{Prim } \mathfrak{B}; \mathfrak{A}, \mathfrak{B})$.

Now, let $\phi \in CP(\text{Prim } \mathfrak{B}; \mathfrak{A}, \mathfrak{B})$, $U \in \mathbb{O}(\text{Prim } \mathfrak{E})$ and $a \in \mathfrak{A}_{\mathfrak{E}}(U) = p(\mathfrak{E}(U))$. As $\sigma(\mathfrak{E}(U)) \subset \mathcal{M}(\mathfrak{B}, \mathfrak{B}(U))$ it follows that $\tau(a) \in \mathcal{Q}(\mathfrak{B}, \mathfrak{B}(U)) = \mathcal{Q}(\mathfrak{B}, \mathfrak{B}(U \cap U_{\mathfrak{B}}))$ and thus $a \in \mathfrak{A}_{\mathfrak{B}}(U \cap U_{\mathfrak{B}})$. Hence $\phi(a) \in \mathfrak{B}(U \cap U_{\mathfrak{B}}) = \mathfrak{B}(U)$, so $\phi \in CP(\text{Prim } \mathfrak{E}; \mathfrak{A}, \mathfrak{B})$. \square

Remark 7.13. Let \mathfrak{A} and \mathfrak{B} be separable C^* -algebras with actions of X . The Kasparov group $KK(X; \mathfrak{A}, \mathfrak{B})$ is constructed exactly as in the classical case, but by *only* considering Kasparov \mathfrak{A} - \mathfrak{B} -modules

$$(E, \phi: \mathfrak{A} \rightarrow \mathbb{B}(E), F)$$

for which the c.p. map

$$a \mapsto \langle x, \phi(a)x \rangle$$

is X -equivariant for every $x \in E$, i.e. ϕ is weakly in $CP(X; \mathfrak{A}, \mathfrak{B})$ by Lemma 2.10. Note that this condition depends *only* on the closed operator convex cone $CP(X; \mathfrak{A}, \mathfrak{B})$. Thus if one has a different space Y which acts on \mathfrak{A} and \mathfrak{B} such that $CP(X; \mathfrak{A}, \mathfrak{B}) = CP(Y; \mathfrak{A}, \mathfrak{B})$, then

$$KK(X; \mathfrak{A}, \mathfrak{B}) = KK(Y; \mathfrak{A}, \mathfrak{B}).$$

In particular, it follows from Lemma 7.12 that if $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ is an extension of separable C^* -algebra, then

$$KK^1(\text{Prim } \mathfrak{B}; \mathfrak{A}, \mathfrak{B}) = KK^1(\text{Prim } \mathfrak{E}; \mathfrak{A}, \mathfrak{B}).$$

Remark 7.14. Recall from Remark 7.8, that we may identify $\text{Prim } \mathfrak{E}$ and $\text{Prim } \mathfrak{A} \sqcup \text{Prim } \mathfrak{B}$ as sets in a canonical way. Thus if $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E}_i \rightarrow \mathfrak{A} \rightarrow 0$ are extensions of C^* -algebras for $i = 1, 2$, then by the above identification, it makes sense to consider the identity map $\text{Prim } \mathfrak{E}_1 = \text{Prim } \mathfrak{E}_2$. Obviously, this map is a homeomorphism if and only if there exists a homeomorphism $\text{Prim } \mathfrak{E}_1 \xrightarrow{\cong} \text{Prim } \mathfrak{E}_2$ which acts as the identity on the canonical subsets $\text{Prim } \mathfrak{A}$ and $\text{Prim } \mathfrak{B}$. So when this is the case, the action of $\text{Prim } \mathfrak{E}_1$ and of $\text{Prim } \mathfrak{E}_2$ on \mathfrak{A} and \mathfrak{B} respectively, are the same.

The following should be thought of as a much weaker criterion than absorbing the zero extension, when one wants to appeal to Theorem 6.11.

Definition 7.15. Let $\mathfrak{e} : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ be an extension of C^* -algebras and $\sigma : \mathfrak{E} \rightarrow \mathcal{M}(\mathfrak{B})$ be the induced $*$ -homomorphism. We will say that \mathfrak{e} is *strongly non-unital* if $\mathfrak{E}/\mathfrak{E}(\mathfrak{J})$ is non-unital for all $\mathfrak{J} \in \mathbb{I}(\mathfrak{B})$ where $\mathfrak{E}(\mathfrak{J}) = \sigma^{-1}(\mathcal{M}(\mathfrak{B}, \mathfrak{J}))$.

Note in particular, that if \mathfrak{A} has no unital quotients, e.g. if \mathfrak{A} is stable, then any extension of \mathfrak{A} is strongly non-unital.

Theorem 7.16. Let $\mathfrak{e}_i : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E}_i \rightarrow \mathfrak{A} \rightarrow 0$ be strongly non-unital extensions of C^* -algebras for $i = 1, 2$. Suppose that \mathfrak{A} is separable and nuclear, that \mathfrak{B} is stable, has finitely many two-sided, closed ideals, which are all σ -unital, and that \mathfrak{B} has a $\text{Prim } \mathfrak{B}$ -tight corona algebra. The following are equivalent.

- (i) \mathfrak{e}_1 and \mathfrak{e}_2 are equivalent,
- (ii) \mathfrak{e}_1 and \mathfrak{e}_2 induce the same action of $\text{Prim } \mathfrak{B}$ on \mathfrak{A} , and

$$[\mathfrak{e}_1] = [\mathfrak{e}_2] \in KK^1(\text{Prim } \mathfrak{B}; \mathfrak{A}, \mathfrak{B}),$$

- (iii) under the canonical identification of $\text{Prim } \mathfrak{E}_i = \text{Prim } \mathfrak{A} \sqcup \text{Prim } \mathfrak{B}$, the identity map $\text{Prim } \mathfrak{E}_1 = \text{Prim } \mathfrak{E}_2$ is a homeomorphism, and

$$[\mathfrak{e}_1] = [\mathfrak{e}_2] \in KK^1(\text{Prim } \mathfrak{E}_1; \mathfrak{A}, \mathfrak{B}).$$

Proof. (i) \Rightarrow (ii): Clear.

(ii) \Rightarrow (i): Suppose that \mathfrak{e}_1 and \mathfrak{e}_2 induce the same action of $\mathsf{X} := \text{Prim } \mathfrak{B}$ on \mathfrak{A} . Since \mathfrak{B} has a tight corona algebra, both extensions are X -full. As in the proof of (i) \Rightarrow (iii) in Theorem 7.5, all two-sided, closed ideals in \mathfrak{B} have the corona factorisation property. Hence by Corollary 6.23 it suffices to show that the extensions are unitisably X -full.

Let τ be the Busby map of \mathfrak{e}_1 . Then \mathfrak{e}_1 is unitisably X -full if and only if $1_{\mathcal{Q}(\mathfrak{B})} - \tau(a)$ is full in $\mathcal{Q}(\mathfrak{B})$ for all $a \in \mathfrak{A}$. Suppose for contradiction that there is an $a \in \mathfrak{A}$ such that $1 - \tau(a)$ is not full in $\mathcal{Q}(\mathfrak{B})$. Since \mathfrak{B} has a tight corona algebra, $1 - \tau(a)$ is full in $\mathcal{Q}(\mathfrak{B}, \mathfrak{J})$ for some proper two-sided, closed ideal \mathfrak{J} in \mathfrak{B} . Lifting $1 - \tau(a)$ to $m \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$, $1 - m$ is a lift of $\tau(a)$. Hence there is an $x \in \mathfrak{E}_1$ with $\sigma(x) = 1 - m$, and $1 - \sigma(x) \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$, where $\sigma : \mathfrak{E}_1 \rightarrow \mathcal{M}(\mathfrak{B})$ is the canonical map. Hence $(1 - x)\mathfrak{E}_1 \subset \mathfrak{E}_1(\mathfrak{J})$. Note that if $x \in \mathfrak{E}_1(\mathfrak{J})$, then $1 = \sigma(x) + m \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ which contradicts $\mathfrak{J} \neq \mathfrak{B}$. In particular, $\mathfrak{E}_1(\mathfrak{J}) \neq \mathfrak{E}_1$ so $\mathfrak{E}_1/\mathfrak{E}_1(\mathfrak{J})$ is non-zero. Now $(1 - x)\mathfrak{E}_1 \subset \mathfrak{E}_1(\mathfrak{J})$ implies that $x + \mathfrak{E}_1(\mathfrak{J})$ is a unit for $\mathfrak{E}_1/\mathfrak{E}_1(\mathfrak{J})$

which is non-zero. This contradicts the strong non-unitality of \mathfrak{e}_1 , and thus \mathfrak{e}_1 , and similarly \mathfrak{e}_2 , is unitisably \mathbf{X} -full.

(ii) \Leftrightarrow (iii): Follows immediately from Lemma 7.12, Remark 7.13 and Remark 7.14. \square

8. APPLICATIONS

8.1. The general machinery. We will use our result in the previous section to classify certain extensions of C^* -algebras. This classifies all real rank zero extensions of AF algebras by strongly classifiable purely infinite algebras. In spirit this means, that given any separable, nuclear, real rank zero C^* -algebra \mathfrak{A} and a two-sided, closed ideal \mathfrak{J} such that $\mathfrak{A}/\mathfrak{J}$ is AF, and \mathfrak{J} is a strongly classifiable purely infinite algebra with finitely many two-sided, closed ideals, then \mathfrak{A} is classified by a K -theoretic invariant.

Notation 8.1. Let \mathbf{Y} be a finite T_0 -space. For any point $y \in \mathbf{Y}$ there is a smallest open set containing y , which we denote by \mathbf{U}^y . Note that $(\mathbf{U}^y)_{y \in \mathbf{Y}}$ is a basis for the topology $\mathbb{O}(\mathbf{Y})$.

There is a partial order on \mathbf{Y} given by $x \leq y :\Leftrightarrow \mathbf{U}^y \subset \mathbf{U}^x$.

We let $\mathbf{Y}\text{-}\mathcal{C}^*\text{-cont}$ denote the category of continuous $\mathbf{Y}\text{-}C^*$ -algebras with \mathbf{Y} -equivariant $*$ -homomorphisms as morphisms.

Similarly, we let $\mathbf{X}\text{-}\mathcal{C}^*\text{-lsc}$ denote the category of lower semicontinuous $\mathbf{X}\text{-}C^*$ -algebras, with \mathbf{X} -equivariant $*$ -homomorphisms as morphisms.

Note that for a finite T_0 -space \mathbf{Y} , the category $\mathbf{Y}\text{-}\mathcal{C}^*\text{-cont}$ is equivalent to the category of C^* -algebras over \mathbf{Y} .

Definition 8.2. Let \mathbf{P} be a finite partially ordered set. We let $\mathbb{Z}\mathbf{P}$ denote the free abelian group generated by elements i_p^q for pairs (p, q) with $p \leq q$, equipped with the bilinear multiplication $i_p^q i_r^s = \delta_{qr} i_p^s$.

We let $\mathfrak{Mod}(\mathbb{Z}\mathbf{P})$ denote the category of $\mathbb{Z}/2$ -graded right $\mathbb{Z}\mathbf{P}$ -modules.

Define the functor $\mathbf{Y}K : \mathbf{Y}\text{-}\mathcal{C}^*\text{-cont} \rightarrow \mathfrak{Mod}(\mathbb{Z}\mathbf{Y}^{\text{op}})$ by

$$\mathbf{Y}K(\mathfrak{A}) = \bigoplus_{y \in \mathbf{Y}} K_*(\mathfrak{A}(\mathbf{U}^y))$$

where i_x^y acts by the map $K_*(\mathfrak{A}(\mathbf{U}^x)) \rightarrow K_*(\mathfrak{A}(\mathbf{U}^y))$ induced by the inclusion $\mathfrak{A}(\mathbf{U}^x) \hookrightarrow \mathfrak{A}(\mathbf{U}^y)$ on the direct summand corresponding to x , and acts as 0 on all other direct summands.

Similarly we define the functor $\mathbb{O}K : \mathbf{X}\text{-}\mathcal{C}^*\text{-lsc} \rightarrow \mathfrak{Mod}(\mathbb{Z}\mathbb{O}(\mathbf{X}))$ by

$$\mathbb{O}K(\mathfrak{A}) = \bigoplus_{\mathbf{U} \in \mathbb{O}(\mathbf{X})} K_*(\mathfrak{A}(\mathbf{U}))$$

where $i_{\mathbf{U}}^{\mathbf{V}}$ acts by the map $K_*(\mathfrak{A}(\mathbf{U})) \rightarrow K_*(\mathfrak{A}(\mathbf{V}))$ induced by the inclusion $\mathfrak{A}(\mathbf{U}) \hookrightarrow \mathfrak{A}(\mathbf{V})$ on the direct summand corresponding to \mathbf{U} , and acts as 0 on all other direct summands.

Definition 8.3. Let \mathbf{X} be a finite T_0 -space and \mathfrak{A} be a lower semicontinuous $\mathbf{X}\text{-}C^*$ -algebra. We say that \mathfrak{A} has *vanishing boundary maps* if the induced map

$$K_*(\mathfrak{A}(\mathbf{U})) \rightarrow K_*(\mathfrak{A})$$

is injective for each open subset \mathbf{U} of \mathbf{X} . If the induced map

$$K_* \left(\sum_{k=1}^n \mathfrak{A}(\mathbf{V}_k) \right) \rightarrow K_*(\mathfrak{A})$$

is injective for any $V_1, \dots, V_n \in \mathbb{O}(X)$, then we say that \mathfrak{A} has *strongly vanishing boundary maps*.

Clearly, if \mathfrak{A} is a continuous Y - C^* -algebra with vanishing boundary maps, then it also has strongly vanishing boundary maps.

Let \mathcal{B} denote the *bootstrap class* in KK -theory, i.e. the class of all separable C^* -algebras satisfying the universal coefficient theorem (UCT) of Rosenberg and Schochet [RS87]. In [MN09] Meyer and Nest introduce an analogues bootstrap class $\mathcal{B}(Y)$ for continuous Y - C^* -algebras. Instead of recalling the definition of $\mathcal{B}(Y)$, we will simply mention, that if \mathfrak{A} is a *nuclear*, continuous Y - C^* -algebra, then \mathfrak{A} is in $\mathcal{B}(Y)$ if and only if $\mathfrak{A}(U^y)$ is in \mathcal{B} for every $y \in Y$.

Recall, that since we are considering finite T_0 -spaces, the notions of continuous Y - C^* -algebras and C^* -algebras over Y are essentially the same. We will need the following result of Bantmann.

Theorem 8.4 ([Ben13]). *Let Y be a finite T_0 -space and \mathfrak{A} and \mathfrak{B} be separable, continuous Y - C^* -algebras. Assume that \mathfrak{A} has vanishing boundary maps and is in $\mathcal{B}(Y)$. Then there is a natural short exact sequence*

$$\mathrm{Ext}_{\mathfrak{Mod}(\mathbb{Z}^{Y^{\mathrm{op}}})}^1(YK(\mathfrak{A}), YK(\mathfrak{B})) \hookrightarrow KK^1(Y; \mathfrak{A}, \mathfrak{B}) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{Mod}(\mathbb{Z}^{Y^{\mathrm{op}}})}(YK(\mathfrak{A}), YK(\mathfrak{B})[1]).$$

We will use this result to obtain a UCT with $\mathbb{O}K$ for lower semicontinuous X - C^* -algebras. The following lemma is what allows us to consider lower semicontinuous X - C^* -algebras instead of continuous Y - C^* -algebras.

Lemma 8.5. *Let X be a finite T_0 -space. Let $Y = \mathbb{O}(X)$ and equip Y with the topology which has the sets*

$$U^V := \{W \in Y : W \subset V\}$$

as a basis, for $V \in Y$. Then U^V is the smallest open set containing V , and there is an isomorphism of categories

$$F : X\text{-}C^*\text{-lsc} \rightarrow Y\text{-}C^*\text{-cont}.$$

Moreover, $Y^{\mathrm{op}} = \mathbb{O}(X)$ as partially ordered sets, and $YK \circ F = \mathbb{O}K$.

Proof. Suppose that $U \in \mathbb{O}(Y)$ is an open set containing $V \in Y$. There are $V_1, \dots, V_n \in Y$ such that $U = \bigcup_{k=1}^n U^{V_k}$. It follows that $V \subset V_k$ for some k , and thus $U^V \subset U^{V_k}$. Hence $U^V \subset U$. Since $V \in U^V$ it follows that U^V is the smallest open subset of Y containing V .

Let \mathfrak{A} be a lower semicontinuous X - C^* -algebra. We let $F(\mathfrak{A})(U^V) = \mathfrak{A}(V)$ for $V \in \mathbb{O}(X)$. Then we have

$$\sum_{V \in Y} F(\mathfrak{A})(U^V) = \sum_{V \in \mathbb{O}(X)} \mathfrak{A}(V) = \mathfrak{A}(X) = \mathfrak{A},$$

and

$$F(\mathfrak{A})(U^V) \cap F(\mathfrak{A})(U^W) = \mathfrak{A}(V) \cap \mathfrak{A}(W) = \mathfrak{A}(V \cap W) = F(\mathfrak{A})(U^{V \cap W}) = \sum_{Z \in U^{V \cap W}} F(\mathfrak{A})(U^Z).$$

for all $V, W \in Y$. Since we clearly have $U^V \cap U^W = U^{V \cap W}$ it follows from [MN09, Lemma 2.35] that $F(\mathfrak{A})$ uniquely determines a continuous Y - C^* -algebra structure by the above assignment. It is easily seen that this induces the desired functor. The functor is an isomorphism with inverse G defined by $G(\mathfrak{A})(V) = \mathfrak{A}(U^V)$.

That $Y^{\text{op}} = \mathbb{O}(X)$ as partially ordered sets, and that $YK \circ F = \mathbb{O}K$, is obvious since U^V is the smallest open set containing V . \square

Remark 8.6. As in the above lemma, it follows that the closed operator convex cones $CP(X; \mathfrak{A}, \mathfrak{B}) = CP(Y; F(\mathfrak{A}), F(\mathfrak{B}))$ are equal. Hence it follows from Remark 7.13 that

$$KK(X; \mathfrak{A}, \mathfrak{B}) = KK(Y; F(\mathfrak{A}), F(\mathfrak{B})).$$

This also shows, that if we construct the category $X\text{-}\mathfrak{K}\mathfrak{K}\text{-}lsc$ with objects being separable, lower semicontinuous X - C^* -algebras and the morphisms are the $KK(X)$ -elements, then this category is isomorphic to the category $\mathfrak{K}\mathfrak{K}(Y)$. Thus we simply define the bootstrap class $\mathcal{B}(X)_{lsc}$, to be the induced bootstrap class. If \mathfrak{A} is nuclear then \mathfrak{A} is in $\mathcal{B}(X)_{lsc}$ if and only if $\mathfrak{A}(U)$ is in \mathcal{B} for every $U \in \mathbb{O}(X)$.

One can of course see, without using the above isomorphism of categories, that $X\text{-}\mathfrak{K}\mathfrak{K}\text{-}lsc$ is triangulated, and that $\mathcal{B}(X)_{lsc}$ is the localising subcategory generated by $i_U(\mathbb{C})$ for all $U \in \mathbb{O}(X)$ (see Section 5.4 for the notation).

Corollary 8.7. *Let X be a finite space and \mathfrak{A} and \mathfrak{B} be separable, lower semicontinuous X - C^* -algebras. Assume that \mathfrak{A} has strongly vanishing boundary maps and belongs to $\mathcal{B}(X)_{lsc}$. Then there is a natural short exact sequence*

$$\text{Ext}_{\mathfrak{M}\mathfrak{O}d(\mathbb{Z}\mathbb{O}(X))}^1(\mathbb{O}K(\mathfrak{A}), \mathbb{O}K(\mathfrak{B})) \hookrightarrow KK^1(X; \mathfrak{A}, \mathfrak{B}) \twoheadrightarrow \text{Hom}_{\mathfrak{M}\mathfrak{O}d(\mathbb{Z}\mathbb{O}(X))}(\mathbb{O}K(\mathfrak{A}), \mathbb{O}K(\mathfrak{B})[1]).$$

Proof. We may assume without loss of generality that X is a T_0 -space. Let F be as in Lemma 8.5. Then $F(\mathfrak{A})$ is in $\mathcal{B}(Y)$ by definition. Note that if $U \in \mathbb{O}(Y) \subset 2^{\mathbb{O}(X)}$, then

$$F(\mathfrak{A})(U) = \sum_{V \in U} F(\mathfrak{A})(U^V) = \sum_{V \in U} \mathfrak{A}(V).$$

Thus, since \mathfrak{A} has strongly vanishing boundary maps, it follows that $F(\mathfrak{A})$ has vanishing boundary maps. The above UCT now follows by combining Theorem 8.4, Lemma 8.5 and Remark 8.6. \square

If $\epsilon : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$ is an X -equivariant extension of lower semicontinuous X - C^* -algebras, then $\mathbb{O}K$ induces a cyclic exact sequence

$$\begin{array}{ccc} & \mathbb{O}K(\mathfrak{E}) & \\ \nearrow & & \searrow \\ \mathbb{O}K(\mathfrak{B}) & \xleftarrow{\circ} & \mathbb{O}K(\mathfrak{A}) \end{array}$$

where the \circ in one arrow indicates that there is a degree shift of the $\mathbb{Z}/2$ -grading. We denote this three-term exact sequence by $\mathbb{O}K_{\Delta}(\epsilon)$. For two such extensions ϵ_1 and ϵ_2 of \mathfrak{A} by \mathfrak{B} then we say that $\mathbb{O}K_{\Delta}(\epsilon_1)$ and $\mathbb{O}K_{\Delta}(\epsilon_2)$ are *congruent*, written $\mathbb{O}K_{\Delta}(\epsilon_1) \equiv \mathbb{O}K_{\Delta}(\epsilon_2)$, if there is a homomorphism $\eta : \mathbb{O}K(\mathfrak{E}_1) \rightarrow \mathbb{O}K(\mathfrak{E}_2)$ such that

$$\begin{array}{ccccc} \mathbb{O}K(\mathfrak{B}) & \longrightarrow & \mathbb{O}K(\mathfrak{E}_1) & \longrightarrow & \mathbb{O}K(\mathfrak{A}) \\ \parallel & & \downarrow \eta & & \parallel \\ \mathbb{O}K(\mathfrak{B}) & \longrightarrow & \mathbb{O}K(\mathfrak{E}_2) & \longrightarrow & \mathbb{O}K(\mathfrak{A}) \end{array}$$

commutes. Note that any η making the diagram commute is an isomorphism by the five lemma.

Proposition 8.8. *Let X be a finite space and let \mathfrak{A} and \mathfrak{B} be separable, lower semicontinuous X - C^* -algebras. Suppose that \mathfrak{A} is an AF algebra, and that $\mathfrak{e}_i : 0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E}_i \rightarrow \mathfrak{A} \rightarrow 0$ for $i = 1, 2$ are X -equivariant extensions. Suppose that \mathfrak{E}_1 and \mathfrak{E}_2 have real rank zero. Then \mathfrak{e}_i are semisplit, and $[\mathfrak{e}_1] = [\mathfrak{e}_2]$ in $KK^1(X; \mathfrak{A}, \mathfrak{B})$ if and only if $\mathbb{O}K_\Delta(\mathfrak{e}_1) \equiv \mathbb{O}K_\Delta(\mathfrak{e}_2)$.*

Moreover, $\mathbb{O}K_\Delta(\mathfrak{e}_i)$ collapses to a short exact sequence

$$0 \rightarrow \mathbb{O}K(\mathfrak{B}) \rightarrow \mathbb{O}K(\mathfrak{E}_i) \rightarrow \mathbb{O}K(\mathfrak{A}) \rightarrow 0$$

for $i = 1, 2$.

Proof. The extensions are semisplit by Corollary 5.31. Clearly $\mathbb{O}K_\Delta(\mathfrak{e}_1) \equiv \mathbb{O}K_\Delta(\mathfrak{e}_2)$ if $[\mathfrak{e}_1] = [\mathfrak{e}_2]$. Since \mathfrak{A} is an AF algebra it easily follows that \mathfrak{A} is in $\mathcal{B}(X)_{lsc}$ and has strongly vanishing boundary maps. Thus by Corollary 8.7 there is a short exact UCT sequence

$$\text{Ext}_{\mathfrak{Mod}(\mathbb{Z}\mathbb{O}(X))}^1(\mathbb{O}K(\mathfrak{A}), \mathbb{O}K(\mathfrak{B})) \hookrightarrow KK^1(X; \mathfrak{A}, \mathfrak{B}) \rightarrow \text{Hom}_{\mathfrak{Mod}(\mathbb{Z}\mathbb{O}(X))}(\mathbb{O}K(\mathfrak{A}), \mathbb{O}K(\mathfrak{B})[1]).$$

Since \mathfrak{E}_i has real rank zero it follows that the induced maps $K_0(\mathfrak{A}(U)) \rightarrow K_1(\mathfrak{B}(U))$ vanish for every $U \in \mathbb{O}(X)$. Moreover, since $K_1(\mathfrak{A}(U)) = 0$ for every U it thus follows that $[\mathfrak{e}_i] \in KK^1(X; \mathfrak{A}, \mathfrak{B})$ induces the zero homomorphism in $\text{Hom}_{\mathfrak{Mod}(\mathbb{Z}\mathbb{O}(X))}(\mathbb{O}K(\mathfrak{A}), \mathbb{O}K(\mathfrak{B})[1])$ for $i = 1, 2$. It follows that $\mathbb{O}K_\Delta(\mathfrak{e}_i)$ collapses to a short exact sequence

$$0 \rightarrow \mathbb{O}K(\mathfrak{B}) \rightarrow \mathbb{O}K(\mathfrak{E}_i) \rightarrow \mathbb{O}K(\mathfrak{A}) \rightarrow 0$$

for $i = 1, 2$. By the above UCT $[\mathfrak{e}_i] \in KK^1(X; \mathfrak{A}, \mathfrak{B})$ is uniquely determined by the induced element in $\text{Ext}_{\mathfrak{Mod}(\mathbb{Z}\mathbb{O}(X))}^1(\mathbb{O}K(\mathfrak{A}), \mathbb{O}K(\mathfrak{B}))$ which is exactly $\mathbb{O}K_\Delta(\mathfrak{e}_i)$. Thus $[\mathfrak{e}_1] = [\mathfrak{e}_2]$ in $KK^1(X)$ if $\mathbb{O}K_\Delta(\mathfrak{e}_1) \equiv \mathbb{O}K_\Delta(\mathfrak{e}_2)$. \square

Definition 8.9. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{A} \rightarrow \mathcal{C}$ be functors. We say that F is a *finer classification invariant* than G if whenever $\phi, \psi: A \rightarrow B$ in \mathcal{A} are isomorphisms such that $F(\phi) = F(\psi)$ then $G(\phi) = G(\psi)$.

Example 8.10. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ and $H: \mathcal{B} \rightarrow \mathcal{C}$ be functors. Then F is a finer classification invariant than $G := H \circ F$.

Recall, that an invariant F *strongly classifies* the objects \mathfrak{A} and \mathfrak{B} , if any isomorphism $F(\mathfrak{A}) \rightarrow F(\mathfrak{B})$ lifts to an isomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$. The following is our main application. In spirit it says that in many cases, if we can separate the finite and infinite part of a C^* -algebra such that the finite part is the quotient, and both ideal and quotient are strongly classified by K -theory, then the C^* -algebra is classified by K -theory.

Theorem 8.11. *Let \mathfrak{A}_i be separable, nuclear, stable C^* -algebras with real rank zero for $i = 1, 2$. Suppose that \mathfrak{J}_i is a closed two-sided ideal in \mathfrak{A}_i such that*

- $X := \text{Prim } \mathfrak{J}_1 \cong \text{Prim } \mathfrak{J}_2$ is finite. Equip \mathfrak{J}_i , \mathfrak{A}_i and $\mathfrak{A}_i/\mathfrak{J}_i$ with the induced X - C^* -algebra structure as in Remark 7.6.
- $\mathfrak{A}_i/\mathfrak{J}_i$ are AF algebras.
- \mathfrak{J}_i has a tight corona algebra (cf. Theorem 7.5).
- There is an invariant F which strongly classifies \mathfrak{J}_1 and \mathfrak{J}_2 , and which is a finer classification invariant than $\mathbb{O}K$.

If there exist isomorphisms

$$\begin{array}{ccccc} F(\mathfrak{J}_1) & \longrightarrow & \mathbb{O}K(\mathfrak{A}_1) & \longrightarrow & \mathbb{O}K^+(\mathfrak{A}_1/\mathfrak{J}_1) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ F(\mathfrak{J}_2) & \longrightarrow & \mathbb{O}K(\mathfrak{A}_2) & \longrightarrow & \mathbb{O}K^+(\mathfrak{A}_2/\mathfrak{J}_2) \end{array}$$

such that the diagram commutes, then $\mathfrak{A}_1 \cong \mathfrak{A}_2$.

Proof. Fix an isomorphism $\phi: \mathfrak{J}_1 \rightarrow \mathfrak{J}_2$ of X - C^* -algebras, which lifts $F(\mathfrak{J}_1) \xrightarrow{\cong} F(\mathfrak{J}_2)$. Note that this isomorphism also induces the given homeomorphism $\text{Prim } \mathfrak{J}_1 \cong \text{Prim } \mathfrak{J}_2$. By Elliott's classification of AF algebras [Ell76] we may also lift $\mathbb{O}K^+(\mathfrak{A}_1/\mathfrak{J}_1) \xrightarrow{\cong} \mathbb{O}K^+(\mathfrak{A}_2/\mathfrak{J}_2)$ to an isomorphism $\psi: \mathfrak{A}_1/\mathfrak{J}_1 \rightarrow \mathfrak{A}_2/\mathfrak{J}_2$ of X - C^* -algebras. Construct the following push-out/pull-back diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{J}_1 & \longrightarrow & \mathfrak{A}_1 & \longrightarrow & \mathfrak{A}_1/\mathfrak{J}_1 \longrightarrow 0 \\ & & \downarrow \cong \phi & & \downarrow \tilde{\phi} & & \parallel \\ \mathfrak{e}_1 : 0 & \longrightarrow & \mathfrak{J}_2 & \longrightarrow & \mathfrak{E}_1 & \longrightarrow & \mathfrak{A}_1/\mathfrak{J}_1 \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ \mathfrak{e}_2 : 0 & \longrightarrow & \mathfrak{J}_2 & \longrightarrow & \mathfrak{E}_2 & \longrightarrow & \mathfrak{A}_1/\mathfrak{J}_1 \longrightarrow 0 \\ & & \parallel & & \downarrow \cong \tilde{\psi} & & \downarrow \cong \psi \\ 0 & \longrightarrow & \mathfrak{J}_2 & \longrightarrow & \mathfrak{A}_2 & \longrightarrow & \mathfrak{A}_2/\mathfrak{J}_2 \longrightarrow 0 \end{array}$$

for which all the rows are short exact sequences. Let $\eta: \mathbb{O}K(\mathfrak{A}_1) \xrightarrow{\cong} \mathbb{O}K(\mathfrak{A}_2)$ be the given isomorphism. Since F and $\mathbb{O}K^+$ are finer classification invariants than $\mathbb{O}K$, it follows that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{O}K(\mathfrak{J}_1) & \longrightarrow & \mathbb{O}K(\mathfrak{E}_1) & \longrightarrow & \mathbb{O}K(\mathfrak{A}_1/\mathfrak{J}_1) \\ \parallel & & \downarrow f & & \parallel \\ \mathbb{O}K(\mathfrak{J}_2) & \longrightarrow & \mathbb{O}K(\mathfrak{E}_2) & \longrightarrow & \mathbb{O}K(\mathfrak{A}_1/\mathfrak{J}_1). \end{array}$$

where $f = \mathbb{O}K(\tilde{\psi}) \circ \eta \circ \mathbb{O}K(\tilde{\phi}^{-1})$. Hence $\mathbb{O}K_{\Delta}(\mathfrak{e}_1) \equiv \mathbb{O}K_{\Delta}(\mathfrak{e}_2)$. By Proposition 8.8 it follows that $[\mathfrak{e}_1] = [\mathfrak{e}_2]$ in $KK^1(\mathsf{X}; \mathfrak{A}_1/\mathfrak{J}_1, \mathfrak{J}_2)$.

Using that ϕ induces the given homeomorphism $\mathsf{X} = \text{Prim } \mathfrak{J}_1 \cong \text{Prim } \mathfrak{J}_2$, it is easily seen that the actions of $\text{Prim } \mathfrak{J}_2$ on $\mathfrak{A}_1/\mathfrak{J}_1$ induced by \mathfrak{e}_1 and \mathfrak{e}_2 , are both equal to the action of X on $\mathfrak{A}_1/\mathfrak{J}_1$ considered earlier composed with the map $\mathbb{O}(\text{Prim } \mathfrak{J}_2) \rightarrow \mathbb{O}(\mathsf{X})$ induced by ϕ^{-1} . Since \mathfrak{J}_2 has a tight corona algebra, it follows from Theorem 7.16 that \mathfrak{e}_1 and \mathfrak{e}_2 are equivalent extensions. Thus $\mathfrak{A}_1 \cong \mathfrak{E}_1 \cong \mathfrak{E}_2 \cong \mathfrak{A}_2$. \square

8.2. Applications to graph C^* -algebras. We will end with an application to graph C^* -algebras. This application says that the class of graph C^* -algebras which contain a purely infinite, two-sided, closed ideal with a finite primitive ideal space, such that the quotient is AF, is classified by a K -theoretic invariant. Moreover, the invariant is computable from the graph.

This generalises a similar result [ERR13b] by the second author, Eilers and Restorff, in which the extensions had to be *full* (in the classical sense). Fullness is a huge limitation on the primitive ideal space of the C^* -algebras for which this is applicable. In this new result there is *no* limitation on the primitive ideal spaces, as long as the two-sided, closed ideal in question has a finite primitive ideal space.

Bentmann and Meyer introduced in [BM14] a new way of classifying (separable) continuous \mathbf{Y} - C^* -algebras up to $KK(\mathbf{Y})$ -equivalence, in the case where the \mathbf{Y} - C^* -algebras have projective dimension at most 2. They did this by adding a particular obstruction class to the invariant. We will vaguely explain the idea behind this. For more details we refer the reader to [BM14].

Let \mathcal{M} be the target category of our invariant F and suppose that \mathcal{M} is *even*, i.e. that \mathcal{M} decomposes as $\mathcal{M}_+ \times \mathcal{M}_-$ with $\mathcal{M}_+[-1] = \mathcal{M}_-$ and $\mathcal{M}_-[-1] = \mathcal{M}_+$. Note that this is the set-up for both of our invariants $\mathbf{Y}K$ and $\mathbb{O}K$. If $F(\mathfrak{A})$ has projective dimension at most 2, then \mathfrak{A} induces an obstruction class $\delta(\mathfrak{A}) \in \text{Ext}_{\mathcal{M}}^2(F(\mathfrak{A}), F(\mathfrak{A})[-1])$. Thus on the class of elements with projective dimension at most 2, we obtain an invariant $F\delta(\mathfrak{A}) = (F(\mathfrak{A}), \delta(\mathfrak{A}))$. An isomorphism $F\delta(\mathfrak{A}) \xrightarrow{\cong} F\delta(\mathfrak{B})$ is an isomorphism $F(\mathfrak{A}) \xrightarrow{\cong} F(\mathfrak{B})$ which intertwines the obstruction classes.

Example 8.12 (The obstruction class for graph C^* -algebras). Let \mathfrak{A} be a lower semi-continuous \mathbf{X} - C^* -algebra which is a graph C^* -algebra $\mathfrak{A} = C^*(E)$, such that every \mathbf{X} -equivariant ideal $\mathfrak{A}(\mathbf{U})$ is invariant under the canonical gauge action γ of \mathbb{T} .

Lemma 8.5 and Remark 8.6 allows us to do everything done in [BM14], for lower semi-continuous \mathbf{X} - C^* -algebras and the invariant $\mathbb{O}K$, instead of continuous \mathbf{Y} - C^* -algebras and the invariant $\mathbf{Y}K$. Thus, as explained in [BM14], $\mathbb{O}K(\mathfrak{A})$ has projective dimension at most 2, and the obstruction class $\delta \in \text{Ext}^2(\mathbb{O}K(\mathfrak{A}), \mathbb{O}K(\mathfrak{A})[-1]) \cong \text{Ext}^2(\mathbb{O}K_0(\mathfrak{A}), \mathbb{O}K_1(\mathfrak{A}))$ is the one induced by the Pimsner–Voiculescu exact sequence

$$\mathbb{O}K_1(\mathfrak{A}) \hookrightarrow \mathbb{O}K_0(\mathfrak{A} \rtimes_{\gamma} \mathbb{T}) \xrightarrow{1-\gamma_*^{-1}} \mathbb{O}K_0(\mathfrak{A} \rtimes_{\gamma} \mathbb{T}) \rightarrow \mathbb{O}K_0(\mathfrak{A}).$$

In [BM14] the proof is carried out only for tight \mathbf{Y} - C^* -algebras, but the case for continuous \mathbf{Y} - C^* -algebras with all \mathbf{Y} -equivariant ideals gauge invariant has the exact same proof.

Consider the graph $E = (E^0, E^1, r, s)$. Every gauge invariant, two-sided, closed ideal in $\mathfrak{A} = C^*(E)$ corresponds to a hereditary and saturated set $H \subset E^0$. Let $H(\mathbf{U})$ denote the hereditary and saturated set corresponding to $\mathfrak{A}(\mathbf{U})$ and write $H(\mathbf{U})_{\text{reg}}$ for $H(\mathbf{U}) \cap E_{\text{reg}}^0$, where E_{reg}^0 is the set of regular vertices. By outsplitting every breaking vertex of all $H(\mathbf{U})$, we may assume that every $H(\mathbf{U})$ has no breaking vertices. Then E induces $\mathbb{Z}\mathbb{O}(X)$ -modules $\mathbb{O}K_0(\mathbb{Z}^{E_{\text{reg}}})$ and $\mathbb{O}K_0(\mathbb{Z}^E)$ by

$$\mathbb{O}K_0(\mathbb{Z}^{E_{\text{reg}}}) := \bigoplus_{\mathbf{U} \in \mathbb{O}(X)} \mathbb{Z}^{H(\mathbf{U})_{\text{reg}}}, \quad \mathbb{O}K_0(\mathbb{Z}^E) := \bigoplus_{\mathbf{U} \in \mathbb{O}(X)} \mathbb{Z}^{H(\mathbf{U})},$$

where the module actions on $\mathbb{O}K_0(\mathbb{Z}^{E_{\text{reg}}})$ and $\mathbb{O}K_0(\mathbb{Z}^E)$ are the obvious ones, i.e. if $\mathbf{U} \subset \mathbf{V}$, then $i_{\mathbf{U}}^{\mathbf{V}}$ acts on the direct summand corresponding to \mathbf{U} by embedding into the direct summand corresponding to \mathbf{V} , and acts as zero on all other direct summands. This is well-defined since $H(\mathbf{U}) \subset H(\mathbf{V})$ whenever $\mathbf{U} \subset \mathbf{V}$. By decomposing the vertices into $E_{\text{reg}}^0 \sqcup E_{\text{sing}}^0$, where E_{sing}^0 is the set of singular vertices, the adjacency matrix of E has the form $\begin{bmatrix} A & \alpha \\ * & * \end{bmatrix}$.

We get a homomorphism $\begin{bmatrix} 1-A^t \\ -\alpha^t \end{bmatrix} : \mathbb{O}K_0(\mathbb{Z}^{E_{\text{reg}}}) \rightarrow \mathbb{O}K_0(\mathbb{Z}^E)$ in the obvious way, and we

let $\mathbb{O}K_0(E)$ (resp. $\mathbb{O}K_1(E)$) denote the cokernel (resp. kernel) of this homomorphism. It is easily verified, as in the classical case when the K -theory of a graph C^* -algebra is computed, that we have a commutative diagram

$$\begin{array}{ccccccc} \mathbb{O}K_1(E) & \longrightarrow & \mathbb{O}K_0(\mathbb{Z}^{E_{\text{reg}}}) & \xrightarrow{\begin{bmatrix} 1-A^t \\ -\alpha^t \end{bmatrix}} & \mathbb{O}K_0(\mathbb{Z}^E) & \longrightarrow & \mathbb{O}K_0(E) \\ \downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong \\ \mathbb{O}K_1(\mathfrak{A}) & \longrightarrow & \mathbb{O}K_0(\mathfrak{A} \rtimes_{\gamma} \mathbb{T}) & \xrightarrow{1-\gamma_*^{-1}} & \mathbb{O}K_0(\mathfrak{A} \rtimes_{\gamma} \mathbb{T}) & \longrightarrow & \mathbb{O}K_0(\mathfrak{A}). \end{array}$$

In particular, the top row determines the obstruction class $\delta(\mathfrak{A}) \in \text{Ext}^2(\mathbb{O}K(\mathfrak{A}), \mathbb{O}K(\mathfrak{A})[-1])$ when identified with $\text{Ext}^2(\mathbb{O}K_0(E), \mathbb{O}K_1(E))$.

We can now give our main application to graph C^* -algebras. Note that the invariant in question can easily be computed from the above method.

Theorem 8.13. *Let \mathfrak{A}_i be graph C^* -algebras for $i = 1, 2$. Suppose that \mathfrak{J}_i is a purely infinite, two-sided, closed ideal in \mathfrak{A}_i , such that $\mathfrak{A}_i/\mathfrak{J}_i$ is AF. Suppose that $\mathsf{X} := \text{Prim } \mathfrak{J}_1 \cong \text{Prim } \mathfrak{J}_2$ is finite. Equip \mathfrak{J}_i , \mathfrak{A}_i and $\mathfrak{A}_i/\mathfrak{J}_i$ with the induced X - C^* -algebra structure as in Remark 7.6. If there exist isomorphisms*

$$\begin{array}{ccccc} \mathbb{O}K\delta(\mathfrak{J}_1) & \longrightarrow & \mathbb{O}K(\mathfrak{A}_1) & \longrightarrow & \mathbb{O}K^+(\mathfrak{A}_1/\mathfrak{J}_1) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathbb{O}K\delta(\mathfrak{J}_2) & \longrightarrow & \mathbb{O}K(\mathfrak{A}_2) & \longrightarrow & \mathbb{O}K^+(\mathfrak{A}_2/\mathfrak{J}_2) \end{array}$$

such that the diagram commutes, then $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$.

Proof. Without loss of generality we may assume that \mathfrak{A}_i is stable. It is well known to experts of graph C^* -algebras, that \mathfrak{A}_i will have real rank zero, since it contains a purely infinite, two-sided, closed ideal with an AF quotient. Moreover, it is well-known that \mathfrak{A}_i is separable and nuclear, and that \mathfrak{J}_i is itself a graph C^* -algebra. By Theorem 7.5 \mathfrak{J}_i has a tight corona algebra. Clearly $\mathbb{O}K\delta$ is a finer classification invariant than $\mathbb{O}K$ so by Theorem 8.11 it suffices to show that $\mathbb{O}K\delta$ classifies \mathfrak{J}_1 and \mathfrak{J}_2 strongly.

It is proven in [BM14, Section 5.3], that for tight Y - C^* -algebras which are graph C^* -algebras, an isomorphism of the invariant $\mathsf{Y}K\delta$ lifts to a $KK(\mathsf{Y})$ -equivalence. The proof in the continuous (not necessarily tight) case is identical, if we furthermore assume that all Y -equivariant ideals are gauge invariant. This is automatic in the tight case, since we assume that Y is finite.

Let $\mathsf{Y} = \mathbb{O}(\mathsf{X})$ and F be as in Lemma 8.5. Then $F(\mathfrak{J}_i)$ are continuous Y - C^* -algebras which are graph C^* -algebras in which every two-sided, closed ideal is gauge invariant. Thus the isomorphism

$$\mathbb{O}K\delta(\mathfrak{J}_1) = \mathsf{Y}K\delta(F(\mathfrak{J}_1)) \xrightarrow{\cong} \mathsf{Y}K\delta(F(\mathfrak{J}_2)) = \mathbb{O}K\delta(\mathfrak{J}_2)$$

lifts to a $KK(\mathsf{Y})$ -equivalence $\alpha \in KK(\mathsf{Y}; F(\mathfrak{J}_1), F(\mathfrak{J}_2))$. Since $KK(\mathsf{Y}; F(\mathfrak{C}), F(\mathfrak{D})) = KK(\mathsf{X}; \mathfrak{C}, \mathfrak{D})$ for all separable, lower semicontinuous X - C^* -algebras \mathfrak{C} and \mathfrak{D} , it follows that α is also a $KK(\mathsf{X})$ -equivalence which lifts $\mathbb{O}K\delta(\mathfrak{J}_1) \xrightarrow{\cong} \mathbb{O}K\delta(\mathfrak{J}_2)$. It follows from

Kirchberg's classification [Kir00] that there is an isomorphism ϕ of X - C^* -algebras which lifts α . Hence $\mathbb{O}K\delta$ classifies \mathfrak{J}_1 and \mathfrak{J}_2 strongly. \square

Remark 8.14 (Classification with filtered K -theory). So far, the classification of non-simple graph C^* -algebras has usually used the invariant *ordered filtered K -theory* FK_Y^+ (also known as ordered filtrated K -theory). Let \mathfrak{A}_i and \mathfrak{J}_i be as in Theorem 8.13 such that $Y := \text{Prim } \mathfrak{A}_1 \cong \text{Prim } \mathfrak{A}_2$ is *finite*. We can obtain classification with ordered filtered K -theory *if* we furthermore assume that \mathfrak{J}_1 and \mathfrak{J}_2 are classified strongly by filtered K -theory FK_X . In fact, suppose that $FK_Y^+(\mathfrak{A}_1) \cong FK_Y^+(\mathfrak{A}_2)$. Since the ordered filtered K -theory FK_Y^+ contains *all* natural transformations of the ordered K -theory of all subquotients, there is an induced diagram

$$\begin{array}{ccccc} FK_X(\mathfrak{J}_1) & \longrightarrow & \mathbb{O}K(\mathfrak{A}_1) & \longrightarrow & \mathbb{O}K^+(\mathfrak{A}_1/\mathfrak{J}_1) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ FK_X(\mathfrak{J}_2) & \longrightarrow & \mathbb{O}K(\mathfrak{A}_2) & \longrightarrow & \mathbb{O}K^+(\mathfrak{A}_2/\mathfrak{J}_2), \end{array}$$

where $\mathbb{O}K$ is the invariant on lower semicontinuous X - C^* -algebras (not Y - C^* -algebras). Since FK_X is a finer classification invariant than $\mathbb{O}K$, it follows from Theorem 8.11 that $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$.

8.3. A final remark. It is possible to obtain classification results similar to Theorem 8.11 with much more general, two-sided, closed ideals than purely infinite ones and not necessarily AF quotients. To obtain these one should apply Corollary 6.23 instead of Theorem 7.16, and thus require that the extensions are X -full. If one wants to use the same method, one should have an invariant G , and a universal coefficient theorem for separable, lower semicontinuous X - C^* -algebras with respect to G . Moreover, one needs to require that the homomorphism induced by the $KK^1(X)$ -class of the extension vanishes, so that G_Δ determines the $KK^1(X)$ -class uniquely, as done in the proof of Proposition 8.8. In the classical case where $G = K_*$, which was done by Rørdam [Rør97], one did not need vanishing of this homomorphism. However, Rørdam uses that the kernel and cokernel of a map on the invariant can be realised by C^* -algebras. It is not obvious whether or not this can be done with the invariant $\mathbb{O}K$.

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